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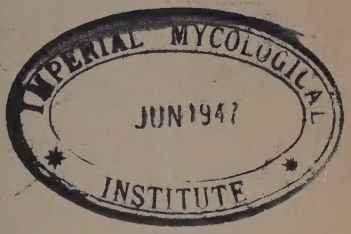
## (DOKLADY)

## DE L'ACADÉMIE DES SCIENCES DE L'URSS

NOUVELLE SÉRIE

1945




VOLUME XLVII

№ 4

ÉDITION DE L'ACADÉMIE DES SCIENCES DE L'URSS
MOSCOU

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NOUVELLE SÉRIE

13-me année

Paraissant tous les dix jours

1945

VOLUME XLVII, № 4

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Édition de l'Académie des Sciences de l'URSS
Moscou 1945

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*MATHEMATICS*

# ISOPERIMETRIC INEQUALITIES FOR CURVED SURFACES

By A. D. ALEXANDROFF

*(Communicated by I. M. Vinogradov, Member of the Academy, 9.X.1944)*

1. Our aim is to obtain estimates for the possible maxima of the area of a geodesic polygon with the sides of given lengths, and of a domain bounded by a curve of a given length, both lying on a surface with the bounded Gaussian curvature, namely, the estimates depending on the upper bound of the curvature in the polygon or in the domain, respectively. It seems worth mentioning that on a surface (even when it is complete and has bounded curvature) the maximum of the area may be attained neither by a geodesic polygon with prescribed sides, nor by a domain with prescribed perimeter. Our Theorems 1 and 3 show that, *e. g.*, such is the case on the hyperbolic paraboloid.

In order to formulate our results we introduce some definitions. Under a domain we shall understand a closed domain homeomorphic to a circle, in which there is given a line element $ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2$ $(EG - F^2 > 0)$ with the coefficients continuous with their first derivatives everywhere in the closed domain. The boundary of the domain is supposed to have a finite length — «the perimeter of the domain». Lengths, angles and areas are defined by means of the given line element. Wether the domain is imbedded in any surrounding space is of no importance.

By a polygon is to be understood a domain, the boundary of which consists of a finite number of geodesic segments.

We say that the curvature in the domain $D$ is $\leqslant K (\geqslant K, = K)$ if for every geodesic triangle lying in $D$ the ratio of its spheric excess $(\alpha + \beta + \gamma - \pi)$ to its area is $\leqslant K (\geqslant K, = K)$.

The term: «Lobachevskian plane» (abbreviated L. p.) will denote the plane where the Lobachevski's geometry takes place; L. p. is entirely defined by its curvature. The Euclidian plane is considered as particular example of L. p.

Theorem 1. *If in a polygon $P$ the curvature is $\leqslant K \leqslant 0$, then the area of $P$ does not exceed that of the polygon $P_0$ which has the sides of the same lengths and is inscribable in a circle on the L. p. with the curvature $K$. And if the area of $P$ is equal to that of $P_0$, $P$ is isometric to $P_0$.*

Theorem 2. *If in a polygon $P$ the curvature does not exceed $K > 0$ and if*: a) *the perimeter of $P$ is $\leqslant 2\pi/\sqrt{K}$ and* b) *no two points of $P$ can be joined by two different geodesics, then the area of $P$ does not exceed that of the polygon $P_0$ which has the sides of the same length and is inscribable in a circle on the hemisphere with the curvature $K$. And if the area of $P$ is equal to that of $P_0$, $P$ is isometric to $P_0$.*

Theorem 3. *If in a domain $D$ the curvature does not exceed $K \leqslant 0$, then the area of $D$ does not exceed that of the circle $D_0$ on the L. p. with the cur-*

*vature $K$, having the same perimeter. And if the area of $D$ is equal to that of $D_0$, $D$ is isometric to $D_0$.*

Theorem 4. *If in a domain $D$ the curvature does not exceed $K>0$ and if:* a) *the perimeter of $D$ is $\leqslant 2\pi/\sqrt{K}$ and* b) *no two points of $D$ can be joined by two different geodesics, then the area of $D$ does not exceed that of the circle $D_0$ on the hemisphere of the curvature $K$, which has the same perimeter. And if the area of $D$ is equal to that of $D_0$, $D$ is isometric to $D_0$.*

The conditions a), b) of Theorems 2 and 4 are essential, for if one of them is omitted, both Theorems break down.

2. Now we shall outline the proof of Theorem 1. After a well-known proposition of the calculus of variations a domain bounded on the Lobachevskian plane by a given segment and a curve of a given length has the greatest area if and only if the curve has a constant curvature. Hence on a simple reasoning we have

Lemma A. The sides of a polygon $P$ with the curvature everywhere equal to $K\leqslant 0$ being given, the greatest area is enclosed by $P$ if and only if it is inscribable in a curve of constant curvature on the L. p. with the curvature $K$.

Owing to this lemma we have only to prove

Lemma B. If in a polygon $P$ the curvature does not exceed $K\leqslant 0$ but is not equal to $K$, then there exists a polygon $P_0$ whose sides are, respectiively, of the same length, the curvature everywhere equal to $K$, and the area greater than that of $P$.

Lemma B will be obtained as a corollary of a set of further Lemmas.

Lemma 1. Let there be given on a L. p. a triangle $ABC$ and two other triangles $A_1B_1D_1$, $A_2C_2D_2$ the sides of which are connected with those of $ABC$ by the equalities: $A_1B_1=AB$, $A_2C_2=AC$, $B_1D_1+D_2C_2=BC$. Then, the sum of the angles $\hat{D}_1$ and $\hat{D}_2$ of the triangles $A_1B_1D_1$ and $A_2C_2D_2$ being $>180°$, the area of the triangle $ABC$ is greater than the sum of the areas of the triangles $A_1B_1D_1$ and $A_2C_2D_2$.

Demonstration. Superposing the triangles $A_1B_1D_1$, $A_2C_2D_2$ upon the triangle $ABC$ so as to make $A_1B_1$ and $A_2C_2$ coincide with $AB$ and $AC$, respectively, we shall find the triangles $A_1B_1D_1$, $A_2C_2D_2$ to be comprised in the triangle $ABC$.

Lemma 2. If in a triangle $T$ the curvature does not exceed $K\leqslant 0$, the angles of this triangle do not exceed the corresponding angles of the triangle $T^0$ on L. p. of curvature $K$, with sides respectively equal in length. And if one of the angles of $T$ is equal to the corresponding angle of $T^0$, then $T$ is isometric to $T^0$ (further, such a triangle $T^0$ will be called «the triangle $T^0$ corresponding to $T$»).

This lemma is known for the case where in $T$ an everywhere continuous Gaussian curvature is in existence (1). Under our somewhat more general suppositions the proof is analogous.

Lemma 3. The curvature in a triangle $T=ABC$ being $\leqslant K\leqslant 0$, the area of $T$ does not exceed that of the triangle $T^0=A^0B^0C^0$ corresponding to $T$.

Demonstration. Drawing in $ABC$ the median $AD$, we divide it into triangles $T_1$ and $T_2$. Construct on the L. p. the corresponding triangles $T_1^0=A_1^0B_1^0D_1^0$, $T_2^0=A_2^0C_2^0D_2^0$. After Lemma 2, the angles of $T_1^0$ and $T_2^0$ are not less than those of $T_1$ and $T_2$; thus, $-\hat{D}_1^0+\hat{D}_2^0\geqslant 180°$ and $\hat{D}_1^0+\hat{D}_2^0=180°$, if and only if $T$ is isometric to $T^0$. Suppose it is not so, then owing to Lemma 1

$$S(T^0)>S(T_1^0)+S(T_2^0) \tag{1}$$

($S$ denotes the area).

Applying the same construction to the triangles $T_1$ and $T_2$, we get the triangles $T_{11}^0$, $T_{12}^0$ and $T_{21}^0$, $T_{22}^0$ satisfying the inequalities:

$$S(T_1^0)\geqslant S(T_{11}^0)+S(T_{12}^0),\ S(T_2^0)\geqslant S(T_{21}^0)+S(T_{22}^0) \tag{2}$$

Proceeding upon this reasoning to the $n$-th step we get the triangles $T_{1\ldots1}^0, \ldots, T_{2\ldots2}^0$ corresponding to the triangles $T_{1\ldots1}, \ldots, T_{2\ldots2}$ into which

the initial triangles $T$ become subdivided. And following the set of inequalities analogous to (1) and (2) we obtain the inequality

$$S(T^0) > S(T^0_{1\ldots1}) + \ldots + S(T^0_{2\ldots2}) \tag{3}$$

But as soon as the triangles $T^0_{1\ldots1}, \ldots, T^0_{2\ldots2}$ are small enough, the sum of their areas differs from the area of the triangle $T$ by an arbitrary small value. Thus, (3) gives: $S(T^0) > S(T)$, q. e. d.

Prove Lemma B. Let the curvature in the polygon $P$ not exceed $K \leqslant 0$. Subdivide $P$ into triangles $T_i$ by means of diagonals and replace each $T_i$ with the corresponding triangle $T^0_i$. The triangles $T^0_i$ form a polygon $P^0$ with the curvature everywhere equal to $K$. After Lemma 3, the area of $P^0$ is greater than that of $P$ excluding the only case where $P$ is isometric to $P^0$.

Lemma B being thus proved, Theorem 1 is proved, too.

The proof of Theorem 2 is just the same, for the conditions a) and b) (they are to be introduced into Lemmas B, 2 and 3) allow to transfer onto the hemisphere all the constructions made on the L. p.

3. The first part of Theorem 3 is an evident consequence of Theorem 1, and it remains only to prove that if the areas of the domain $D$ and of the circle $D_0$ are equal, the curvature in $D$ is equal to $K$.

Suppose the opposite to be true. Then in $D$ there exists a triangle $T$ in which the curvature is not equal to $K$. $T^0$ being the corresponding triangle, Lemma 3 gives

$$\sigma = S(T_0) - S(T) > 0 \tag{4}$$

Construct in $D$ a polygon $P$ subject to the conditions

1) $$S(D) - S(P) < \frac{\sigma}{2} \tag{5}$$

2) $D_1$ being the circle on the L. p. with the perimeter equal to that of $P$,

$$S(D_1) - S(D_0) < \frac{\sigma}{2} \tag{6}$$

3) $P$ contains the triangle $T$.

Divide the polygon $P - T$ into triangles by means of diagonals. $P$ is thereby subdivided also. Replacing each triangle of this subdivision with a corresponding triangle we get a complex of triangles, which we denote with $P_1$.

After Lemma 3 the areas of all the triangles of $P_1$ are not less than the areas of the triangles into which $P$ is subdivided. And as $T$ and $T^0$ are among these triangles

$$S(P_1) - S(P) \geqslant S(T^0) - S(T) = \sigma \tag{7}$$

Hence in virtue of (5)

$$S(P_1) - S(D) > \frac{\sigma}{2} \tag{8}$$

According to Lemma 2, the angles do not decrease when the triangles of $P$ are replaced by those of $P_1$. Therefore the sum of the angles around every inner vertex of the complex $P_1$ is $> 360°$ (for these are the vertices of $T$ only, and the curvature in $T$ is not equal to $K$). We increase the length of all the sides meeting at one of these vertices $O$. The areas of all the triangles around $O$ will then increase, while the angles around $O$ will decrease. From this it is clear that without changing its perimeter $P_1$ can be replaced by such a complex $P_0$

as has the greater area, while the sum of angles around every inner vertex of $P_0$ is $=360°$. This $P_0$ is a polygon in which the curvature is equal to $K$ and

$$S(P_0) \geqslant S(P_1) \tag{9}$$

But $S(P_0) \leqslant S(D_1)$ and therefore $S(P_1) \leqslant S(D_1)$. Now applying (6) and (8) we find that $S(D_0) > S(D)_1$, q. e. d.

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Received
9. X. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*MATHÉMATIQUES*

# SUR L'APPROXIMATION DES FONCTIONS HOLOMORPHES PAR LES FONCTIONS ENTIÈRES

Par M. KELDYCH, membre correspondant de l'Académie

1°. On sait qu'une fonction continue $f(z)$, définie sur une ligne jordannienne $C$ sans points multiples et dont l'origine et l'extrémité sont à l'infini, peut être approchée par une fonction entière $G(z)$ de sorte que l'on ait sur la ligne $C$

$$|f(z)-G(z)|<\varepsilon(|z|)$$

où $\varepsilon(t)$ est une fonction positive arbitraire dont la borne inférieure est non nulle dans chaque intervalle fini et qui décroît aussi vite que l'on veut quand $t\to\infty$. Ce résultat a été établi par Carleman pour le cas d'une ligne jordannienne rectifiable (¹) et peut être étendu au cas d'une ligne jordannienne arbitraire (²). Dans la présente Note nous considérons l'approximation des fonctions holomorphes au moyen des fonctions entières dans les domaines ayant le point à l'infini pour point de la frontière, et nous établissons une série de théorèmes sur l'ordre de décroissance de l'écart pour $|z|\to\infty$, ainsi qu'une série de théorèmes sur la croissance de la fonction entière approximative.

2°. En considérant l'approximation au moyen des fonctions entières, holomorphes à l'intérieur des domaines à frontière jordannienne et continues sur la frontière, on peut établir les propositions suivantes.

Théorème 1. *Soit $C$ une courbe de Jordan sans points multiples dont l'origine et l'extrémité sont à l'infini, $D$ un domaine borné par cette courbe, $f(z)$ une fonction holomorphe dans $D$ et continue dans $D+C$ (sauf le point à l'infini). Quels que soient les nombres positifs $\varepsilon$ et $\eta$, il existe une fonction entière $G(z)$ vérifiant l'inégalité*

$$|f(z)-G(z)|<\varepsilon e^{-|z|^{\frac{1}{2}-\eta}} \qquad (1)$$

*dans le domaine fermé $\overline{D}=D+C$. D'ailleurs, il existe un domaine du type considéré et une fonction $f(z)$ holomorphe dans ce domaine pour laquelle le théorème en question n'est plus vrai, si dans l'inégalité (1) on remplace le second membre par une fonction décroissante quelconque $\psi(|z|)$, pour laquelle*

$$\liminf_{t\to\infty}\frac{\log\psi(t)}{t^{1/2}}=-\infty$$

*et de même dans le cas où $\psi(t)=e^{-at^{\frac{1}{2}}}$, $a>0$.*

Dans certaines classes particulières de domaines on peut réaliser l'approximation par une fonction entière de manière à faire l'écart tendre vers zéro plus vite quand $t\to\infty$.

Théorème 2. *Si $D$ est un domaine du type considéré dans le théorème 1 et situé à l'intérieur de l'angle $|\arg z| \leqslant \alpha/2$, et $f(z)$ est une fonction holomorphe dans $D$ et continue sur $D+C$ (sauf le point à l'infini), alors quels que soient $\varepsilon$ et $\eta$ positifs, on peut construire une fonction entière $G(z)$ vérifiant l'inégalité*

$$|f(z)-G(z)|<\varepsilon e^{-|z|^{\frac{\pi}{\alpha}-\eta}} \tag{2}$$

*Pour les fonctions définies dans un angle, le théorème n'est plus vrai quand on remplace le second membre de l'inégalité (2) par une fonction décroissante pour laquelle*

$$\liminf_{t\to\infty} \frac{\log \psi(t)}{t^{\frac{\pi}{\alpha}}} = -\infty$$

*et de même dans le cas où $\psi(t)=e^{-at^{\frac{\pi}{\alpha}}}$, $a>0$.*

Théorème 3. *Soit $D$ un domaine situé dans la bande $|y|\leqslant\pi/2$, borné par des lignes de Jordan joignant $-\infty$ et $+\infty$, et $f(z)$ une fonction holomorphe dans $D$ et continue dans $D+C$ (sauf aux points à l'infini). Quels que soient $\varepsilon$ et $\eta$, on peut construire une fonction entière vérifiant l'inégalité*

$$|f(z)-G(z)|<\varepsilon e^{-e^{(1-\eta)|z|}} \tag{3}$$

*et d'ailleurs pour les fonctions définies dans une bande le théorème n'est plus vrai, si l'on remplace le second membre de l'inégalité par une fonction décroissante pour laquelle*

$$\liminf_{t\to\infty} \frac{\log \psi(t)}{e^t} = -\infty$$

*et de même dans le cas où $\psi(t)=e^{-ae^t}$, $a>0$.*

3°. Pour les fonctions analytiques dans les domaines fermés (sauf le point à l'infini) on peut dans certains cas établir quelles sont les restrictions que l'on peut imposer à la croissance de la fonction approximative, en tenant compte de la croissance de la fonction $f(z)$ et de la décroissance de l'écart. Nous désignerons par $M(r)$ le maximum du module de la fonction $f(z)$ aux points du domaine $D$ situés à l'intérieur du cercle $|z|\leqslant r$, et par $\mathfrak{M}(r)$ le maximum du module de la fonction entière dans le cercle $|z|\leqslant r$. On a alors les propositions suivantes.

Théorème 4. *Soit $f(z)$ une fonction holomorphe dans l'angle $|\arg z|\leqslant\alpha/2$, $\rho$ un nombre vérifiant l'inégalité $0\leqslant\rho\leqslant\pi/\alpha$, $\varepsilon$ et $\delta$ des nombres positifs arbitraires. Il existe une fonction entière vérifiant l'inégalité*

$$|f(z)-G(z)|<\varepsilon e^{-|z|^{\rho}} \tag{4}$$

*dans l'angle $|\arg z|\leqslant\frac{\alpha}{2}-\delta$, et dont le maximum du module vérifie l'inégalité*

$$\log \mathfrak{M}(r)<(1+r)^{\frac{\pi}{2\pi-\alpha}}\left[K+k\max_{t\leqslant k^{r+1}} \frac{t^{\rho}+\log^{*} M(t)}{(1+t)^{\frac{\pi}{2\pi-\alpha}}}\right]$$

*où $k$ est une constante qui dépend de $\delta$, et $K$ une constante qui dépend de $\varepsilon$ et $\delta$. On peut aussi construire une fonction entière vérifiant l'inégalité (4) dont la croissance est bornée par l'inégalité*

$$\log \mathfrak{M}(r)<(r+\delta)^{\varkappa}\left[K+k(r+\delta)^{\rho}+k\log^{+}M(r+\delta)\right]$$

*où $\varkappa$ est le plus grand des nombres 2, $\pi/(2\pi-\alpha)$.*

Signalons une conséquence de ce théorème: si dans l'angle $|\arg z|\leqslant\alpha/2$ la fonction $f(z)$ est telle que l'on a

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \nu \tag{5}$$

alors pour $0 \leqslant \rho \leqslant \pi/\alpha$ on peut construire une fonction entière vérifiant l'inégalité (4) dans l'angle $|\arg z| \leqslant \frac{\alpha}{2} - \delta$, dont l'ordre ne surpasse pas le plus grand des nombres $\nu$, $\rho$, $\pi/(2\pi - \alpha)$, et d'ailleurs la borne indiquée pour l'ordre ne peut pas être abaissée.

Théorème 5. *Soit $f(z)$ une fonction holomorphe dans la bande $|y| \leqslant \pi/2$, $\varepsilon$ et $\delta$ des nombres positifs arbitraires. Il existe une fonction entière vérifiant l'inégalité*

$$|f(z) - G(z)| < \varepsilon e^{-|z|^\rho} \tag{6}$$

*et telle que*

$$\log \mathfrak{M}(r) < (1 + r(\log^+ r)^2)[K + k \log^+ r + k(r \log^+ r)^\rho + \\ + k \log^+ M(kr \log^+ r + k)]$$

*où $k$ est une constante qui dépend de $\delta$, et $K$ dépend de $\varepsilon$ et $\delta$. On peut aussi construire une fonction entière vérifiant* (6) *et telle que sa croissance est bornée par l'inégalité*

$$\log \mathfrak{M}(r) < (r + \delta)^2 [K + k(r + \delta)^2 + k \log^+ M(r + \delta)]$$

On déduit de ce théorème le corollaire suivant: si une fonction $f(z)$ holomorphe dans la bande $|y| \leqslant \pi/2$ vérifie la condition (5), il existe une fonction entière vérifiant l'inégalité (6) et dont l'ordre ne surpasse pas le plus grand des deux nombres $\rho + 1$, $\nu + 1$. La borne indiquée pour l'ordre ne peut pas être abaissée.

Théorème 6. *Soit $f(z)$ une fonction holomorphe dans la bande $|y| \leqslant \pi/2$, $\varepsilon$ et $\delta$ des nombres positifs arbitraires. Il existe une fonction entière vérifiant l'inégalité*

$$|f(z) - G(z)| < \varepsilon e^{-e^{|z|}}$$

*dans la bande $|y| \leqslant \frac{\pi}{2} - \delta$ dont la croissance est bornée par l'inégalité*

$$\log \mathfrak{M}(r) < K[e^{(1+\delta)r} + (r + \delta)^2 \log^+ M(r + \delta)]$$

4°. Si $f(z)$ est une fonction d'une variable réelle possédant la première dérivée, on peut évaluer la croissance de la fonction entière approximative en fonctions de la croissance de cette dérivée et de la décroissance de l'écart à l'infini. Nous ne signalons ici qu'un seul résultat particulier:

Théorème 7. *Supposons que $f(z)$ est définie pour $-\infty < x < +\infty$, et $\mu(t)$ est le maximum de $|f'(x)|$ pour $|x| < t$. Si*

$$\limsup_{t \to \infty} \frac{\log \mu(t)}{\log t} = \nu$$

*il existe une fonction entière dont l'ordre ne surpasse pas $\nu + 1$, qui vérifie l'inégalité*

$$|f(x) - G(x)| < \varepsilon$$

*où $\varepsilon$ est un nombre positif arbitraire. L'ordre de croissance de $G(x)$ indiqué ne peut pas être abaissé.*

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Manuscrit reçu
le 11. I. 1945.

LITTÉRATURE CITÉE

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*MATHEMATICS*

## CHARACTERISTIC CYCLES

By L. PONTRJAGIN, Corresponding Member of the Academy

In his papers ([1],[2]) H. Whitney introduced the notion of the fibre-bundle and investigated it chiefly in the case where the fibres are spheres. My paper ([3]) deals essentially with the same object, but in a particular case, namely, with tangents to an oriented manifold. The methods used are stronger than those due to Whitney in the sense that I constructed more cycles than Whitney did. In the present note I show that my methods are applicable to the general case. The following definitions 1 and 2 are due to Whitney, only a few remarks are added.

Definition 1. Suppose $A$ is a topological space with a given topological group $\Gamma$ of homeomorphisms, $B$ is a complex consisting of simplices $T_1, T_2, \ldots$, and $P$ is a topological space. We shall say that $P$ is a fibre-bundle or a skew product of $A$ and $B$, $P = P(A, B)$, if the following conditions are fulfilled:

1) To every point $y \in B$ there corresponds a subset $A_y \subset P$; the subsets $A_y$ and $A_z$ corresponding to distinct points $y \in B$ and $z \in B$ do not overlap.

2) To every simplex $T_\alpha$ there corresponds a homeomorphic mapping $\xi_\alpha$ of the topological product $A \cdot T_\alpha$, *i. e.* a function $\xi_\alpha(x, y) = \xi_{\alpha y}(x)$ is defined $(x \in A,\ y \in T\alpha)$ such that $\xi_{\alpha y}$ is a homeomorphic mapping of the space $A$ onto $A_y$.

3) If $y \in T_\alpha \cap T_\beta$, then $\xi_{\alpha y}^{-1} \xi_{\beta y} = \eta_{\alpha\beta y} \in \Gamma$, and $\eta_{\alpha\beta y}$ gives a continuous mapping of $T_\alpha \cap T_\beta$ into the group $\Gamma$.

It is evident that if $y \in T_\alpha \cap T_\beta \cap T_\gamma$, then

$$\eta_{\alpha\beta y}\, \eta_{\beta\gamma y} = \eta_{\alpha\gamma y} \tag{1}$$

Suppose all the simplices $T_1, T_2, \ldots$ to be oriented and denote by $\varepsilon_{\alpha\beta}$ the coefficient with which $T_\beta$ enters the boundary of the simplex $T_\alpha$. If $\varepsilon_{\alpha\beta} \neq 0$, we put $\eta_{\alpha\beta y} = \zeta_{\alpha\beta y}$. Let $\varepsilon_{\alpha\beta} \neq 0$, $\varepsilon_{\alpha\gamma} \neq 0$, $\varepsilon_{\beta\delta} \neq 0$, $\varepsilon_{\gamma\delta} \neq 0$; then (1) implies for $y \in T_\delta$

$$\zeta_{\alpha\beta y}\, \zeta_{\beta\delta y} = \zeta_{\alpha\gamma y}\, \zeta_{\gamma\delta y} \tag{2}$$

If we put $\Delta_\Gamma T_\alpha = \sum_\beta \varepsilon_{\alpha\beta}\, \zeta_{\alpha\beta y}\, T_\beta$, then (2) can be represented as $\Delta_\Gamma \Delta_\Gamma T_\alpha = 0$. It is easy to verify that, given $\zeta_{\alpha\beta y}$ satisfying (2), $\eta_{\alpha\beta y}$ satisfying the relation (1) are determined uniquely. It is easy to prove the existence theorem:

A. Suppose $A$ is a topological space with a given topological group $\Gamma$ of its homeomorphism, $B$ is a complex consisting of the simplices $T_1, T_2, \ldots$ with $\Delta T_\alpha = \sum_\beta \varepsilon_{\alpha\beta}\, T_\beta$, and $\zeta_{\alpha\beta y} \in \Gamma$ are continuous functions of $y \in T_\beta$ satisfying

the relations (2), defined for $\alpha$, $\beta$ with $\varepsilon_{\alpha\beta} \neq 0$. There exists then a skew product $P(A, B)$ for which $\xi_{xy}^{-1} \xi_{\beta y} = \zeta_{\alpha\beta y}$.

Definition 2. Let $P = P(A, B)$ and $P' = P'(A, B)$ be skew products with the same group $\Gamma$. A homeomorphic mapping $f$ of the space $P'$ onto the space $P$ will be called isomorphic mapping of the skew product $P'(A, B)$ onto the skew product $P(A, B)$ if $f(A'_y) = A_y$, and $\xi_{xy}^{-1} f \xi'_{xy} = \tau_{xy} \in \Gamma$ for $y \in T_\alpha$. It follows immediately from the last equality that

$$\zeta_{\alpha\beta y} = \tau_{\alpha y}^{-1} \zeta'_{\alpha\beta y} \tau_{\beta y} \tag{3}$$

for $y \in T_\beta$. The following theorem can be easily proved:

B. The skew products $P(A, B)$ and $P'(A, B)$ with the determining functions $\zeta_{\alpha\beta y}$ and $\zeta'_{\alpha\beta y}$ are isomorphic if and only if there exist continuous functions $\tau_{\alpha y} \in \Gamma$ $(y \in T_\alpha)$ satisfying the conditions (3).

Whitney noted that if $P(A, B)$ is a skew product and $f$ is a continuous mapping of a complex $C$ into $B$, then a skew product $P(A, C, f)$ of $A$ and $C$ arises having the same group $\Gamma$ as the initial product $P(A, B)$; thereby $P(A, C, f)$ is determined uniquely up to isomorphisms. It is easy to prove that if $f$ and $g$ are equivalent, then $P(A, C, f)$ and $P(A, C, g)$ are isomorphic. It turns out, however, that the isomorphism of the products $P(A, C, f)$ and $P(A, C, g)$ often implies the equivalence of the mappings $f$ and $g$. This leads us to the following

Definition 3. A skew product $P(A, B)$ with the group $\Gamma$ is said to be $n$-dimensionally universal for the space $A$ with the group $\Gamma$ if the following conditions are fulfilled:

1) A skew product $P(A, C)$ with the group $\Gamma$ where $C$ is of dimension less than or equal to $n$, is always isomorphic to $P(A, C, f)$, where $f$ is a continuous mapping of $C$ into $B$.

2) The skew products $P(A, C, f)$ and $P(A, C, g)$ are isomorphic if and only if the mappings $f$ and $g$ of the complex $C$ into $B$ are equivalent.

Definition 4. Preserving the notations of definition 3, suppose that $B$ is a $b$-dimensional oriented manifold, and denote by $z$ a $(b-r)$-dimensional cycle from $B$. Suppose $f$ is a continuous mapping of $C$ into $B$ and $T$ is any $r$-dimensional oriented simplex from $C$. The intersection index of $f(T)$ with $z$ is denoted by $u_z^r(T)$. It is easy to see that $u_z^r$ is an $r$-dimensional $\nabla$-cycle from $C$, the homology class of which is uniquely determined by that of the cycle $z$ and by the class of the mapping $f$. Thus, since $P(A, B)$ is universal, the $\nabla$-cycle $u_z^r$ is an invariant of the skew product $P(A, C, f)$. We shall call $u_z^r$ characteristic cycle of the skew product $P(A, C, f)$.

Let $H_l = H(k, l)$ be the manifold of all $k$-dimensional oriented planes in the $(k+l)$-dimensional Euclidean space $R^{k+l}$ passing through the origin of $R^{k+l}$. Let now $A$ be a $k$-dimensional oriented Euclidean space $R^k$, while the group $\Gamma$ is the group of all isometric mappings of $R^k$ onto itself that preserve the origin and the orientation in $R^k$. Let us define the skew product $P(R^k, H_l)$: a point of the space $P$ is a pair $(R_y^k, x)$ where $R_y^k \in H_l$, $x \in R_y^k$; then the set of all the pairs $(R_y^k, x)$ with the fixed $R_y^k$ is a fibre $A_y$. Whitney pointed out that $P(R^k, H_l)$ satisfies the condition 1) of the definition 3, when $l = n$. In Theorem 1 I prove this assertion in a different way showing moreover that $P(R^k, H_l)$ is universal.

Theorem 1. *The skew product $P(R^k, H_l)$ is $(l-1)$-dimensionally universal for the space $R^k$ with the group of rotations $\Gamma$.*

Proof. Let us show that $P(R^k, H_l)$ is universal, when $l$ is sufficiently large, the dimension can be diminished without difficulties. Suppose $C$ is the complex consisting of the simplices $T_1, T_2, \ldots$, and $C^r$ is the subcomplex of $C$ consisting of the simplices of dimension less than or equal to $r$. Let $P(R^k, C)$ be a skew product with the determining functions $\zeta_{\alpha\beta y} \in \Gamma$, $y \in T_\beta$ (see A). Let us construct a continuous mapping $f$ of the complex $C$ into $H_l$ such that $P(R^k, C, f)$ be isomorphic to $P(R^k, C)$, $f(y) = \bar{R}_y^k \in H_l$. Simulta-

neously we shall construct an isometric mapping $\xi_{\alpha y}(y \in T_\alpha)$ of the space $R^k$ onto the space $\bar{R}^k_y$ in such a way that $\bar{\xi}^{-1}_{\alpha y}\bar{\xi}_{\beta y} = \zeta_{\alpha\beta y}$. Assume $f$ and $\bar{\xi}_{\alpha y}$ to be already constructed for $C^r$, $f(y) = \bar{R}^k_y \subset R^{k+l}$, when $y \in C^r$. We suppose $R^{k+l} \subset \subset R^{2k+l}$ and $\bar{R}^k$ is the orthogonal complement to $R^{k+l}$ in $R^{2k+l}$. Let $T_\alpha$ be an arbitrary $(r+1)$-dimensional simplex from $C$ with a centre $p$. Put $f(p) = R^k$ and define an isometric mapping $\bar{\xi}_{\alpha p}$ of the space $R^k$ onto $\bar{R}^k$ in an arbitrary way. Let $S_\alpha$ be the boundary of the simplex $T_\alpha$, then the position of a point $z \in T_\alpha$ in the simplex $T_\alpha$ can be determined by the pair $(y, t)$, where $y \in S_\alpha$, and $t$ is a real parameter: $0 \leqslant t \leqslant \pi/2$, $(y, 0) = p$, $(y, \pi/2) = y$, $z = (y, t)$. If $T_\beta$ is an arbitrary $r$-dimensional face of the simplex $T_\alpha$, then $\bar{\xi}_{\beta y}$ is defined. We define the function $\bar{\xi}_{\alpha z}(z = (y, t),\ y \in T_\beta)$ by putting

$$\bar{\xi}_{\alpha z}(x) = \bar{\xi}_{\alpha p}(x)\cos(t) + \bar{\xi}_{\beta y}(\zeta^{-1}_{\alpha\beta y}(x))\sin(t),\quad x \in R^k \tag{4}$$

The function $f$ is defined as follows:

$$f(z) = \bar{\xi}_{\alpha z}(R^k). \tag{5}$$

If $y$ belongs to the $r$-dimensional faces $T_\beta$ and $T_\gamma$ of the simplex $T_\alpha$, then, putting $T_\delta = T_\beta \cap T_\gamma$, we see, in virtue of (2), that the relation (4) defines the mapping $\bar{\xi}_{\alpha z}$ independently of the choice of the $r$-dimensional face of $T_\alpha$.

Now let $f$ and $g$ be continuous mappings of $C$ into $H_l$ such that $P(R^k, C, f)$, and $P(R^k, C, g)$ are isomorphic. This means that there exists an isometric mapping $h_y$ of the plane $f(y)$ onto the plane $g(y)$ that depends continuously on $y \in C$. Suppose $R^{k+l} \subset R^{2k+2l}$, $\bar{R}^{k+l}$ is the orthogonal complement to $R^{k+l}$ in $R^{2k+2l}$, and $\varphi$ is an isometric mapping of $R^{k+l}$ onto $\bar{R}^{k+l}$. Then $\varphi g$ is a continuous mapping of $C$ into $H_{k+2l}$ which is isotopic to $g$, because $\varphi$ can be realized by rotating $R^{k+l}$ continuously in $R^{2k+2l}$. Take for $x \in f(y)$

$$\psi_{yt}(x) = \varphi(h_y(x))\cos(t) + x\sin(t),\quad \psi_t(y) = \psi_{ty}(f(y)) \tag{6}$$

It is evident that $\psi_t(y)$ is a $k$-dimensional plane in $R^{2k+2l}$, and therefore $\psi$ is a mapping of $C$ into $H_{k+2l}$. We have from (6)

$$\psi_0 = \varphi g,\quad \psi_{\frac{\pi}{2}} = f$$

$g$ and $\varphi g$ being equivalent, $f$ and $g$ are equivalent in $H_{k+2l}$. Theorem 1 is thus proved.

Homologies in the manifold $H(k, l)$ are studied enough in my paper [3]. The following theorem is useful for the study of homotopical properties of $H(k, l)$.

Theorem 2. *For $n \leqslant l-1$ the $n$-dimensional homotopical group of the manifold $H(k, l)$ is isomorphic to the $(n-1)$-dimensional homotopical group of the manifold $\Gamma_k$, where $\Gamma_k$ is the group of positive rotations of the $k$-dimensional Euclidean space.*

Proof. The proof will be carried through for sufficiently large $l$, as in Theorem 1. Let $R^k_0$ and $R^k_1$ be two mutually orthogonal $k$-dimensional subspaces of $R^{k+l}$. Suppose that the group $\Gamma_k$ operates in $R^k_1$. Denote by $\varphi$ an isometric mapping of $R^k_1$ onto $R^k_0$. Let further $S^{n-1}$ be the $(n-1)$-dimensional oriented sphere and $\psi_y$ be an element of the group $\Gamma_k$, depending continuously on $y \in S^{n-1}$. For $x \in R^k_1$ we put

$$\theta(x, y, t) = \varphi(x)\cos(t) + \psi_y(x)\sin(t);\quad \theta(y, t) = \theta(R^k_1, y, t). \tag{7}$$

It is evident that $\theta(y, t)$ is a $k$-dimensional plane in $R^{k+l}$, and $\theta(y, 0) = R^k_0$,

$\theta\left(y, \frac{\pi}{2}\right) = R_1^k$. Thus, $\theta(y, t)$ realizes a continuous mapping of the direct product $S^{n-1} \cdot T$ into $H_l$, where $T$ is the segment $0 \leqslant t \leqslant \pi/2$. Thereby the bases $S^{n-1} \cdot 0$ and $S^{n-1} \cdot \pi/2$ of the cylinder $S^{n-1} \cdot T$ are carried into single points. Consequently, we have a continuous mapping of the sphere $S^n$ into $H_l$. This correspondence between the mapping of $S^{n-1}$ into $\Gamma_k$ and the mapping of $S^n$ into $H_l$ proves to give an isomorphism of the corresponding groups of homotypes. The proof can be carried out approximately in the same way, as in Theorem 1.

It is worth noting that the construction of universal skew products presents no great difficulty for any compact Lie group $\Gamma$, so that $P(R^k, H_l)$ is not an exception in this respect.

Received
9. II. 1945.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*HYDRODYNAMICS*

# UNSTEADY FILTRATION WITH A FREE SURFACE

By L. A. GALIN

*(Communicated by L. S. Leibenson, Member of the Academy, 9.VII.1944)*

1. The case of a Fluid Having no Weight. If the filtration takes place in a horizontal layer and the force of gravity is perpendicular to the direction of motion, then the fluid may be considered as having no weight. This occurs in oil filtration.

At the initial moment the oil occupies the domain between the contours $\Gamma(t_0)$ and $\Gamma_2$, the pressure on which is respectively $p_1$ and $p_2$; $\Gamma_2$ is a circumference of a small radius $\delta$ (a well). At the instant $t$ this domain will be situated between the contours $\Gamma(t)$ and $\Gamma_2$, the pressures on which will be also $p_1$ and $p_2$ (the pressures are constant during the process of filtration. These conditions of the problem are due to L. S. Leibenson (1)).

The function representing the unit circle $|\zeta| < 1$ on the simply connected domain bounded by $\Gamma(t)$ is $z(\zeta, t)$; we also have $z(t, 0) = 0$. The filtration velocities are determined by the velocity potential as

$$v_x = \partial\varphi/\partial x, \quad v_y = \partial\varphi/\partial y \tag{1}$$

where $\varphi = -\frac{k}{\rho g} p$.

Herein $k$ is the filtration coefficient, and $\rho$, the density of the fluid (2).

The complex potential in the $\zeta$-plane is

$$w(\zeta) = \frac{-(p_2 - p_1)\, k}{\lg \delta_1 \rho g} \lg \zeta - \frac{p_1 k}{\rho g} = \varphi^* + i\psi^* \tag{2}$$

where $\delta_1 = \delta \left| \partial\zeta/\partial z \right|_{\zeta=0}$ is the radius of the circumference corresponding in the $\zeta$-plane to the contour $\Gamma_2$.

The displacement velocity of a point of the unit circle in the $\zeta$-plane

$$v = -\frac{k(p_2 - p_1)}{m\rho g \lg \delta_1} \left| \frac{\partial\zeta}{\partial z} \right|^2_{\zeta = e^{i\vartheta}} \tag{3}$$

The displacement of a point of the unit circumference in a time $dt$

$$\varepsilon = -\frac{k(p_2 - p_1)}{m\rho g \lg \delta_1} \left| \frac{\partial\zeta}{\partial z} \right|^2_{\zeta = e^{i\vartheta}} dt$$

Here $m$ is the porosity of the medium. After a time $dt$ we obtain in the $\zeta$-domain a contour which is close to a circumference. It may be shown that the function representing the domain bounded by this contour on a circle is

$$\zeta_1(\zeta) = \zeta - \zeta S_\gamma \left[ A \left| \frac{\partial \zeta}{\partial z} \right|^2_{\zeta = e^{i\vartheta}} dt \right] \qquad (4)$$

where $A = -k(p_2 - p_1)/m\rho g \lg \delta_1$, and $S_\gamma$ is Schwarz's symbol for the circle. But then

$$z(t + dt, \zeta) = z(\zeta_1(\zeta)) = z\left(\zeta - \zeta S_\gamma \left[ A \left| \frac{\partial \zeta}{\partial z} \right|^2_{\zeta = e^{i\vartheta}} dt \right]\right)$$

and, consequently,

$$\frac{\partial z}{\partial t} = -\frac{\partial z}{\partial \zeta} \zeta S_\gamma \left[ A \left| \frac{\partial \zeta}{\partial z} \right|_{\zeta = e^{i\vartheta}} \right]$$

Hence

$$-\frac{\partial z}{\partial t} \frac{1}{\zeta} : \frac{\partial z}{\partial \zeta} = S_\gamma \left[ A \left| \frac{\partial \zeta}{\partial z} \right|^2_{\zeta = e^{i\vartheta}} \right] \qquad (5)$$

Determining the real part of this expression on the unit circumference, we shall have

$$\mathrm{Re}\left[ -\frac{\partial z}{\partial t} \frac{1}{\zeta} : \frac{\partial z}{\partial \zeta} \right]_{\zeta = e^{i\vartheta}} = -\left( \frac{k(p_2 - p_1)}{\rho g m} \right) : \left[ \lg \delta - \lg \left| \frac{\partial z}{\partial \zeta} \right|_{\zeta = 0} \right]$$

Hence follows that

$$\mathrm{Re}\left[ \frac{\partial z}{\partial t} \zeta \frac{\partial z}{\partial \zeta} \right]_{\zeta = e^{i\vartheta}} = \left( \frac{k(p_2 - p_1)}{\rho g m} \right) : \left[ \lg \delta - \lg \left| \frac{\partial z}{\partial \zeta} \right|_{\zeta = 0} \right] \qquad (6)$$

This equation together with the initial condition $z(0, \zeta) = z_0(\zeta)$ determines the function $z(\zeta, t)$.

2. The Case of Heavy Fluid. When the motion of the fluid takes place in a vertical plane, the forces of gravity must be taken into account. The problem considered below corresponds to the case of a flow to a drain.

Instead of equation (1) we shall have here [2]

$$\varphi^* = -k\left( \frac{p^*}{\rho g} + y \right) \qquad (7)$$

The pressure $p$ is a harmonic function satisfying the conditions similar to those which are satisfied by $\varphi^*$ in the preceding problem.

Thus, by analogy with (3), we find

$$-\left( \frac{\rho g}{k} \frac{\partial \varphi}{\partial n} + \rho g \frac{\partial y}{\partial n} \right) = \frac{p_2 - p_1}{\lg \delta_1} : \left| \frac{\partial z}{\partial \zeta} \right|_{\zeta = e^{i\vartheta}} \qquad (8)$$

or for the displacement of a point of the unit circumference

$$\varepsilon_1 = -\frac{k}{m} \left\{ \frac{p_2 - p_1}{\rho g \lg \delta_1} - \mathrm{Re}\left[ i\zeta \frac{\partial z}{\partial \zeta} \right]_{\zeta = e^{i\vartheta}} \right\} dt : \left| \frac{\partial z}{\partial \zeta} \right|^2_{\zeta = e^{i\vartheta}}$$

Hence finally

$$\operatorname{Re}\left[\frac{\partial z}{\partial t}\zeta\frac{\partial z}{\partial\zeta}-i\frac{k}{m}\zeta\frac{\partial z}{\partial\zeta}\right]_{\zeta=e^{i\vartheta}}=\frac{k(p_2-p_1)}{\rho g m}:\left(\lg\delta-\lg\left|\frac{\partial z}{\partial\zeta}\right|_{\zeta=0}\right) \tag{9}$$

The initial condition is: $z(0,\zeta)=z_0(\zeta)$.

3. The Solution of the Problem in the Case of a Fluid Having no Weight when $z_0(\zeta)$ is a Polynomial. Introduce into equation (6) the new variable

$$\tau=\int_0^t\frac{k(p_2-p_1)}{m\rho g}\,dt:\left(\lg\delta-\lg\left|\frac{\partial z}{\partial\zeta}\right|_{\zeta=0}\right) \tag{10}$$

Then we obtain

$$\operatorname{Re}\left[\frac{\partial z}{\partial\tau}\zeta\frac{\partial z}{\partial\zeta}\right]_{\zeta=e^{i\vartheta}}=1. \tag{11}$$

If the problem is solved and we know

$$\left(\frac{\partial z}{\partial\zeta}\right)_{\zeta=0}=F(\tau)$$

then for the determination of $\tau$ we obtain a differential equation with separable variables

$$\frac{d\tau}{dt}=-\frac{k(p_2-p_1)}{m\rho g}:(\lg F(\tau)-\lg\delta)$$

From the equation (11) all physical parameters are eliminated, which considerably simplifies the computations.

If $z_0(\zeta)$ is a polynomial of degree $m$, let us assume that

$$z(\tau,\zeta)=\sum_{n=1}^{m}\alpha_n(\tau)\zeta^n \tag{12}$$

Substituting in (11) this expression, and putting the coefficient of $\zeta^0$ at unity, and the coefficients of $\zeta^n$ at zero for $n\neq 0$, we obtain a system of non-linear differential equations

$$\begin{gathered}
\alpha_1(\tau)\,\alpha_m'(\tau)+m\alpha_m(\tau)\,\alpha_1'(\tau)=0\\
2\alpha_2(\tau)\,\alpha_m(\tau)+\alpha_1(\tau)\,\alpha_{m-1}'(\tau)+m\alpha_m(\tau)\,\alpha_2'(\tau)+(m-1)\,\alpha_{m-1}(\tau)\,\alpha_1'(\tau)=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
m\alpha_m(\tau)\,\alpha_m'(\tau)+(m-1)\,\alpha_{m-1}(\tau)\,\alpha_{m-1}'(\tau)+\ \ldots\ +3\alpha_3(\tau)\,\alpha_3'(\tau)+\\
+2\alpha_2(\tau)\,\alpha_2'(\tau)+\alpha\ (\tau)\,\alpha_1'(\tau)=1.
\end{gathered}$$

Solving them with respect to the derivatives, we obtain

$$\left.\begin{gathered}
\alpha_1'(\tau)=A_1(\alpha_1(\tau),\ \alpha_2(\tau),\ \ldots,\ \alpha_m(\tau))\\
\alpha_2'(\tau)=A_2(\alpha_1(\tau),\ \alpha_2(\tau),\ \ldots,\ \alpha_m(\tau))\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\alpha_m'(\tau)=A_m(\alpha_1(\tau),\ \alpha_2(\tau),\ \ldots,\ \alpha_m(\tau))
\end{gathered}\right\} \tag{13}$$

The initial conditions are

$$\tau=0:\quad \alpha_1(\tau)=\alpha_1^0,\ \ \alpha_2(\tau)=\alpha_2^0,\ \ldots,\ \alpha_m(\tau)=\alpha_m^0$$

The $A_1, A_2, \ldots, A_m$ are rational functions of the $\alpha_n(t)$.

The obtained system may be solved by the usual methods. It may be shown that the degree of the polynomial (12) can be no greater than $m$,

since this would contradict the uniqueness of the solution of the system of differential equations in a domain where the solution must obviously be unique ([3]).

At the instant that a point of the contour attains the well $|z(t, \zeta)|_{\min} = 0$ and consequently

$$\frac{|z(t, \zeta)|_{\max}}{|z(t, \zeta)|_{\min}} = \infty \tag{14}$$

Suppose that at this moment $z(t, \zeta)$ is a simple function. Let $z(0, \zeta) = z_0(\zeta)$ be a polynomial of degree $m$, and consequently

$$z(t, \zeta) = \sum_{n=1}^{m} a_n(t)\, \zeta^n$$

Then, by Littlewood's theorem ([4]) $|a_n(t)| < en\,|a_1(t)|$, and consequently

$$|z(t, \zeta)|_{\max} < |a_1(t)|\, e \sum_{n=1}^{m} n = |a_1(t)|\, e\, \frac{m^2 + m}{2}$$

On the other hand, by Koebe's theorem ([4])

$$|z(t, \zeta)|_{\min} > \frac{1}{4}\,|a_1(t)|$$

Thus

$$\frac{|z(t, \zeta)|_{\max}}{|z(t, \zeta)|_{\min}} < 2e(m^2 + m)$$

which contradicts the condition (14), and it follows that at the instant that a point of the contour attains the well $z(t, \zeta)$ cannot be simple. But $z(0, \zeta) = z_0(\zeta)$ is simple. Consequently, the loss of simpleness, which is accompanied by the formation of a point of return on the contour of the domain, occurs before this instant. Up to the moment that this point of return arises, the obtained solution will be valid.

The function satisfying the equation (5) (as well as the function satisfying the equation (9)) and the condition $z(0, \zeta) = z_0(\zeta)$ may also be obtained in the form

$$z(t, \zeta) = z_0(\zeta) + z_1(\zeta)\, t + z_2(\zeta)\, t^2 + \dots \tag{15}$$

Substituting (15) into (5) and putting $t = 0$, we obtain the expression for $z_1(\zeta)$ determined through $z_0(\zeta)$.

Substituting the values of $z_0(\zeta)$ and $z_1(\zeta)$ into (5) differentiated with respect to $t$, and putting $t = 0$, we obtain the expression for $z_2(\zeta)$ through $z_0(\zeta)$ and $z_1(\zeta)$ a. s. o. It may be shown that in the case where $z_0(\zeta)$ is a rational function, all the $z_n(\zeta)$ will be also rational functions of $\zeta$.

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Received
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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*HYDRODYNAMICS*

# ON THE DISPLACEMENT OF THE OIL-BEARING CONTOUR

By P. J. POLUBARINOVA-KOTSCHINA

*(Communicated by L. S. Leibenson, Member of the Academy, 9.VII.1944)*

1. In many oil deposits the inflow of oil to the wells occurs under the pressure of the edge waters surrounding the oil-bearing domain. Leibenson (1) has proposed the following problem: In a porous medium is given a plane domain occupied by a fluid, the contour of which at the initial moment is known. Inside the domain there are wells. To determine the displacement of the contour of the domain, if it be assumed that the given values of pressure at the contour of the domain and at the well contours remain constant all the time. This condition is equivalent to the assumption that the outside of the domain is occupied by a fluid of zero viscosity.

As usual, we suppose the filtration velocity to have a potential

$$u = m\frac{dx}{dt} = \frac{\partial \varphi}{dx}, \quad v = m\frac{dy}{dt} = \frac{\partial \varphi}{dy}, \quad \varphi = -\frac{k}{\mu} p(x, y, t) \tag{1}$$

where $p$ is the pressure of the fluid; $k$ is the permeability of the ground; $\mu$ is the viscosity of the fluid; $m$ is the porosity of the ground; $t$ is time.

The contour of the domain is described by the equation

$$\varphi(x, y, t) = -\frac{k}{\mu} p_0 = \text{const} \tag{2}$$

Differentiating this equation with respect to $t$, we find a condition that must be satisfied on the contour all along the movement

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 = -m\frac{\partial \varphi}{\partial t} \tag{3}$$

On an auxiliary $\zeta$-plane consider $n$ equidistant wells situated inside the unit circle on the circumference $r = |\zeta_0| < 1$. The complex potential of the current will be:

$$\omega(\zeta) = -\frac{Q}{2\pi} \lg \frac{\zeta^n - \zeta_0^n}{1 - \overline{\zeta_0^n}\zeta^n} \tag{4}$$

Let us represent the circular domain of the $\zeta$-plane on a certain domain of the $z$-plane by means of the equation

$$z = f(\zeta) = A_1\zeta + A_2\zeta^2 + \dots \tag{5}$$

The current in the $z$-plane will be characterized by the function (4), and for the production rate $Q$ of the well we shall have:

$$Q=\frac{2\pi(p_0-p_1)k}{\mu\lg\frac{(1-\zeta_0^n\bar{\zeta}_0^n)\,|f'(\zeta_0)|}{n\delta|\zeta_0|^{n-1}}}=\frac{2\pi k\Delta pq}{\mu\lg\frac{(1-c^{2n})f'(c)}{n\delta c^{n-1}}}=M_0q \tag{6}$$

where $\delta$ is the radius of the well; $c=\zeta_0$ for $t=0$; $p_0$ and $p_1$ are the pressure at the contour of the domain and the contour of the well, respectively.

We consider $A_1, A_2, \ldots$ as given for $t=0$ and proceed to determine $A_1, A_2, \ldots$ as functions of $t$.

For $f(\zeta)$ we find by (3) the equation

$$\bar{\zeta}\dot{f}(\zeta)f'(\bar{\zeta})+\zeta\dot{f}(\bar{\zeta})f'(\zeta)=-q\frac{1-\zeta_0^n\bar{\zeta}_0^n}{\left(1-\frac{\zeta_0^n}{\zeta^n}\right)(1-\bar{\zeta}_0^n\zeta^n)} \tag{7}$$

The accent denotes the derivative by $\zeta$, and the point denotes the derivative by the absolute time $\tau$ connected with $t$ as follows:

$$\tau=\frac{M_0n}{\pi m}t=\frac{2kn\Delta p}{\mu m\lg\frac{(1-c^{2n})f'(c)}{n\delta c^{n-1}}}t \tag{8}$$

We suppose $\zeta_0$ and $A_k$ to be real, which corresponds to the case of a symmetric contour. Putting in (5) $\zeta=e^{i\varphi}$, we obtain from (6):

$$\begin{gathered}2\mathrm{Re}\,(\dot{A}_1e^{i\varphi}+\dot{A}_2e^{2i\varphi}+\ldots)(A_1e^{-i\varphi}+2A_2e^{-2i\varphi}+\ldots)=\\=-q(1+2\zeta_0^n\cos n\varphi+2\zeta_0^{2n}\cos 2n\varphi+\ldots)\end{gathered} \tag{9}$$

whence we find the system of equations for the $A_k$:

$$\begin{gathered}A_1\dot{A}_1+2A_2\dot{A}_2+3A_3\dot{A}_3+\ldots=-q/2\\A_1\dot{A}_2+2A_2\dot{A}_3+\ldots+2A_2\dot{A}_1+3A_3\dot{A}_2+\ldots=0\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\A_1\dot{A}_{n+1}+\ldots+(n+1)A_{n+1}\dot{A}_1+\ldots=-\zeta_0^nq\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\A_1\dot{A}_{2n+1}+\ldots+(2n+1)A_{2n+1}\dot{A}_1+\ldots=-\zeta_0^{2n}q\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{gathered} \tag{10}$$

2. The unit circumference as the initial contour. Let the wells be situated at the distance $c<1$ from the centre of the circle. Then, for $t=0$ $A_1=1$, $A_2=A_3=\ldots=0$, and from the system (10) we find that for $t=0$

$$\dot{A}_1=-1/2,\quad \dot{A}_{n+1}=-c^n,\ \dot{A}_{kn+1}=-c^{kn},\ \ldots(k=1,2,\ldots) \tag{11}$$

and all other $\dot{A}_k$ are equal to zero.

Differentiating (10) with respect to $t$, we find for $t=0$:

$$\ddot{A}_1 = -\frac{1}{4(1-c^{2n})^2}\left\{1-3c^{4n}+(4n+2)c^{2n}+\frac{n}{2\gamma}(1+4c^{2n}+c^{4n})\right\}$$

$$\ddot{A}_{n+1} = -\frac{c^n}{(1-c^{2n})^2}\left\{(n+1)(1-c^{4n})+2nc^{2n}+\frac{n}{2\gamma}(1+4c^{2n}+c^{4n})\right\}$$

$$\ddot{A}_{2n+1} = -\frac{c^{2n}}{(1-c^{2n})^2}\left\{(2n+1)(1-c^{4n})+2nc^{2n}+\frac{n}{2\gamma}(1+4c^{2n}+c^{4n})\right\} \qquad (12)$$

. . . . . .

$$\gamma = \lg\frac{1-c^{2n}}{nc^{n-1}}$$

The third derivatives have a more involved form. If, however, the number of wells is large, the quantities $c^{2n}$, $c^{4n}$, ... are small in comparison with unity, and we may neglect them, which considerably simplifies the formulae.

For the point $x_0$ nearest to the well we have:

$$x_0 = A_1 + A_{n+1} + \ldots + A_{2n+1} + \ldots = 1 + \alpha_1\tau + \alpha_2\tau^2 + \alpha_3\tau^3 + \ldots \qquad (13)$$

$$\alpha_1 = -\frac{1}{2}\frac{1+c^n}{1-c^n},$$

$$\alpha_2 = -\frac{1+(4n+2)c^n+4nc^{2n}+(4n-2)c^{3n}-c^{4n}+\frac{n}{\gamma}(1+4c^{2n}+c^{4n})}{8(1-c^{2n})(1-c^n)^2} \qquad (14)$$

$$\alpha_3 = -\frac{P_1(c)+\frac{1}{\gamma}P_2(c)+\frac{3n^2}{\gamma^2}(1+4c^{2n}+c^{4n})^2}{48(1-c^n)^5(1+c^n)^3}$$

$$P_1(c) = 3+(6n^2+18n+18)c^n+(36n^2+60n+12)c^{2n}+$$

$$+(90n^2+82n+2)c^{3n}+(120n^2-30)c^{4n}+(90n^2-34n-26)c^{5n}+$$

$$+(36n^2-60n+12)c^{6n}+(6n^2-18n+6)c^{7n}+3c^{8n}$$

$$P_2(c) = 5n+(10n^2+8n)c^n+(36n^2+27n+1)c^{2n}+(50n^2+28n)c^{3n}+$$

$$+(108n^2-n+1)c^{4n}+(50n^2-8n)c^{5n}+(36n^2-27n-1)c^{6n}+$$

$$+(10n^2-4n)c^{7n}-(4n+1)c^{8n}$$

The radius-vector of the point with the slowest displacement is obtained by substituting $c$ with $-c$ in the expression for $x_0$.

The variation of the production rate with time is defined by the formula:

$$Q = Q_0\left[1+\frac{n(1+4c^{2n}+c^{4n})}{2\gamma(1-c^{2n})^2}\tau+\frac{P(c)+\frac{3n^2}{\gamma}(1+4c^{2n}+c^{4n})^2}{8\gamma(1-c^{2n})^2}\tau^2+\ldots\right] \qquad (15)$$

$$P(c) = 2n+(24n^2+15n+1)c^{2n}+(60n^2-n+1)c^{4n}+$$

$$+(24n^2-15n-1)c^{6n}-(n+1)c^{8n}$$

For a single well ($n=1$) we may write $f(\zeta)$ as follows:

$$z=f(\zeta)=\zeta-\frac{\tau}{2}\frac{\zeta(1+c\zeta)}{1-c\zeta}-\frac{\tau^2\zeta}{8(1-c^2)^2}\left\{1+6c^2-3c^4+\right.$$

$$\left.+\frac{8c(1+c^2-c^4)-4c^2(1+2c^2-c^4)\zeta^2}{(1-c\zeta)^2}+\frac{(1+4c^2+c^4)(1+c\zeta)}{\gamma(1-c\zeta)}\right\}+\ldots \qquad (16)$$

The point $\zeta_0$ moves towards the contour. At the moment that the outward contour touches the wells $\zeta_0=1$. For $n=1$, $\zeta_0=0$ (in this case $\zeta_0$ is all along equal to zero) the formula (7) yields the equation obtained by Galin ([2]).

3. A cardioid as the initial contour. Let its equation be

$$z=\zeta+a\zeta^2, \quad \zeta=e^{i\varphi}$$

We suppose that $a<{}^1/_2$ lest the curve should possess a double point. To the well $z=z_0$ taken on the $x$-axis there corresponds $\zeta=\zeta_0$; for $t=0$ we have $\zeta_0=c$. The system (10) yields:

$$\begin{aligned} &\dot{A}_1+2a\dot{A}_2=-{}^1/_2, \quad 2a\dot{A}_1+\dot{A}_2+2a\dot{A}_3=-c \\ &\dot{A}_k+2a\dot{A}_{k+1}=-c^{k-1} \quad (k=1, 2, \ldots) \end{aligned} \qquad (17)$$

One of the quantities, for instance $\dot{A}_1$, remains arbitrary. There exists a unique value

$$\dot{A}_1=-\frac{1-2ac}{2(1+2ac)(1-4a^2)}$$

for which the obtained series are convergent. Then

$$\dot{A}_2=-\frac{1}{4a}+\frac{1-2ac}{4a(1+2ac)(1-4a^2)}, \quad \dot{A}_k=-\frac{c^k}{1+2ac} \quad (k=3, 4, \ldots)$$

For a well at the point $z_0=0$ $(c=0)$ the solution has the form of a polynomial

$$z=A_1(t)\zeta+A_2(t)\zeta^2$$

$A_1(t)$ and $A_2(t)$ satisfy the system of equations:

$$A_1\dot{A}_1+2A_2\dot{A}_2=\frac{\lg\delta}{\lg(A_1:\delta)}, \quad A_1\dot{A}_2+2A_2\dot{A}_1=0 \qquad (18)$$

The integration of the system yields

$$\tau=1+2a^2+\frac{1-a^2}{2\lg\frac{1}{\delta}}-\frac{1}{\lg\frac{1}{\delta}}\left(A_1^2+\frac{2a^2}{A_1^4}\right)\lg\frac{A_1}{\delta}+$$

$$+\frac{1}{2\lg\frac{1}{\delta}}\left(A_1^2-\frac{a^2}{A_1^4}\right), \quad A_2=\frac{a}{A_1^2} \qquad (19)$$

Before the contour of the domain approaches the well, there will be an instant $\tau$ at which a point of return appears on the cardioid. This will occur for $A_1=2A_2=(2a)^{1/3}$. The solution of (19) is only true up to the instant. The

same will occur (cf. (2)) with polynomials of any degree. The reason probably is that we neglect the inertial terms in Darcy's equations, which is not legitimate for great velocities.

4. A well in a half-plane. The condition at the contour of the domain may be written in the form

$$\frac{d\omega}{d\zeta}\frac{d\bar{\omega}}{d\bar{\zeta}} - \frac{m}{2}\left[\frac{d\omega}{d\zeta}\dot{f}(\zeta)\bar{f}'(\bar{\zeta}) + \frac{d\bar{\omega}}{d\bar{\zeta}}\dot{\bar{f}}(\bar{\zeta})f'(\zeta)\right] = 0 \tag{20}$$

or, since on the contour $\zeta = \bar{\zeta} = \xi$, $\frac{d\bar{\omega}}{d\bar{\xi}} = -\frac{d\omega}{d\xi}$

$$\frac{d\omega}{d\xi} + \frac{m}{2}\left[\dot{f}(\xi)\bar{f}'(\xi) - \dot{\bar{f}}(\xi)f'(\xi)\right] = 0 \tag{21}$$

For a single well

$$\omega = -\frac{q}{2\pi}\lg\frac{\zeta - \eta_0 i}{\zeta + \eta_0 i}$$

Putting

$$z = f(\zeta) = \zeta + A(\zeta)t + B(\zeta)t^2 + C(\zeta)t^3 + \ldots,$$

we find

$$A(\zeta) = -\frac{q_0}{\pi m}\frac{1}{\zeta + ci}$$

$$B(\zeta) = -\frac{q_0^2(1+\varepsilon)}{4\pi^2 m^2 c}\frac{1}{\zeta + ci} - \frac{q_0^2}{2m^2\pi^2 c}\frac{1}{(\zeta + ci)^2}$$

$$C(\zeta) = -\frac{q_0^3(4 + 6\varepsilon + 3\varepsilon^2)}{24m^3\pi^3 c^4}\frac{1}{\zeta + ci} - \frac{q_0^3(7 + 5\varepsilon)i}{24\pi^3 m^3 c^3}\frac{1}{(\zeta + ci)^2} -$$

$$-\frac{q_0^3}{12\pi^3 m^3 c^3}\frac{1}{(\zeta + ci)^3}, \quad \varepsilon = \frac{1}{\lg(2c/\delta)}, \quad c = \eta_0\big|_{t=0}$$

In a similar way may be treated problems on motion in vertical plane, when gravity is to be taken into account. Equation (3) will then include a term with the first degree of $\partial\varphi/\partial y$, and equation (20) will contain additional terms.

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Received
9.VII.1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
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*THEORY OF ELASTICITY*

# ON THE PROPAGATION VELOCITY OF DISCONTINUITIES OF THE DISPLACEMENT DERIVATIVES ON THE SURFACE OF AN UNHOMOGENEOUS ELASTIC BODY OF ARBITRARY FORM

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1. Considering the time $t$ as a spatial coordinate, suppose that the displacements $u$, $v$, $w$ which are usually considered in the theory of elasticity, are defined inside and on the boundary of a cylinder $C$ having its generatrices parallel to the $t$-axis, and its base formed by the elastic body under consideration. For the sake of brevity we put $u=u_1$, $v=u_2$, $w=u_3$. We shall assume that in the neighbourhood of its point $M(x^0, y^0, z^0)$ the body is bounded by the surface $z=f(x, y)$, where the function $f$ has continuous derivatives up to the $(n+2)$-nd order inclusively, or, for brevity, will be said to have a smoothness of order $n+2$. We shall at first assume that $n \geqslant 3$.

We shall consider, in general, an unhomogeneous, anisotropic elastic body free from the action of external forces; we shall assume that all coefficients in the equations of elasticity and in the boundary conditions have a smoothness of order $n$, respectively, $n+1$, and do not depend on time.

Suppose that the $u_i$ have in the neighbourhood of the $M^0(t^0, x^0, y^0, z^0)$ a smoothness of order $n$, and that on the surface of $C$ in the neighbourhood of this point the $u_i$ possess continuous derivatives of the $(n+1)$-st order everywhere, except on a certain two-dimensional surface $S_2$ having a smoothness of order $n+2$ and passing through $M^0$, on which they have singularities of a kind to be described below. Our problem is to determine the inclination of $S_2$ at the point $M^0$.

2. Let us choose in the neighbourhood of $M^0$ some transformation of $t$, $x$, $y$, $z$ having a smoothness of order $n+2$ and possessing the following properties:

a) it carries the boundary of the cylinder under consideration into a part of the plane $\Pi_3$, and transforms $S_2$ into a part of the plane $\Pi_2$;

b) the new coordinate surfaces passing through $M^0$ are tangent to old at $M^0$;

c) the new origin is situated at the point $M^0$;

d) at the point $M^0$ we have $\frac{\partial t'}{\partial t} = \frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y} = \frac{\partial z'}{\partial z} = 1$, where the new coordinates are marked by an accent.

Let us pass in the equations of the theory of elasticity, referred to hereafter as (1), to these new coordinates. We shall find that the $u_i$ in the neighbourhood of $M^0$ satisfy a system (1′) of three homogeneous linear equations of the second order, wherein the coefficients of the partial derivatives preserve at the point $M^0$ the values they possessed in the original system (1). Similarly the boundary conditions (2) on the free surface of the body will be replaced by new linear equations (2′), in which the coefficients before the first derivatives will have the same values at $M^0$. We will not write out the equations (1), (2), (1′) and (2′).

We will assume that all terms of these equations are written on the left-hand side. The functions $u_i$ will have in the neighbourhood of $M^0$ a smoothness of order $n$. On $\Pi_3$ outside $\Pi_2$ they will have a smoothness of order $n+1$.

Let us suppose that on $\Pi_3$ in the neighbourhood of $M^0$ the functions $u_i$ have the form

$$u_i(t', x', y', 0) =$$

$$= M_i^{(n+1)}(t', x', y') + a_i(t', x', y')[\tau t' + \alpha x' + \beta y']^{n+1} \ln|\tau t' + \alpha x' + \beta y'| +$$

$$+ b_i(t', x', y')[\tau t' + \alpha x' + \beta y']^n |\tau t' + \alpha x' + \beta y'| + R_i(t', x', y') \qquad (3)$$

$$(i = 1, 2, 3)$$

where the $M_i^{(n+1)}(t', x', y')$ are polynomials in $t', x', y'$ of degree $(n+1)$; the $a_i(t', x', y')$, $b_i(t', x', y')$ and $R_i(t', x', y')$ are functions having a smoothness of order $n+1$; moreover, among the quantities $a_i^0 = a_i(0, 0, 0)$, $b_i^0 = b_i(0, 0, 0)$ at least one is not equal to zero, and

$$D^{(k)}R_i(t', x', y') = 0\{[|t'| + |x'| + |y'|]^{n-k+1}\}, \quad k = 0, 1, \ldots, n+1,$$

where $D^{(k)}R_i$ denotes one of the derivatives of order $k$ of $R_i$; on the plane $\Pi_3$ the equation $\tau t' + \alpha x' = \beta y' = 0$ represents the plane $\Pi_2$.

3. The equations (2) and, consequently, also (2′) may be resolved at the point $M^0$ with respect to $\partial u_i/\partial z$, respectively $\partial u_i/\partial z'$, since the determinant composed of their coefficients is one of the diagonal determinants in the fundamental equations connecting the stresses with the deformations, and all such determinants must be positive, if we are to assume, which we do, that there exists an elastic potential for any homogeneous body with the same elastic properties as those present at some point of the unhomogeneous body under consideration. It is easily seen that the equations (1′) may be resolved at $M^0$ with respect to $\partial^2 u_i/\partial z'^2$. Therefore we may without any difficulty construct polynomials $M_i'(t', x', y', z')$ of degree $(n+1)$ which coincide with $M_i^{(n+1)}(t', x', y')$ for $z' = 0$, and make the left-hand sides of the equations (1′), respectively (2′), vanish at the point $M^0$, as well as their partial derivatives by $t', x', y', z'$, respectively $t', x', y'$, up to the order $n-1$, respectively $n$.

Put in the whole neighbourhood of $M^0$ under consideration

$$t' = \varepsilon t'', \; x' = \varepsilon x'', \; y' = \varepsilon y'', \; z' = \varepsilon z'',$$

$$u_i' = u_i - M_i'(t', x', y', z') = \varepsilon^{n+1} u_i'' - M_i''(t'', x'', y'', z'')\,\varepsilon^{n+1} \ln \varepsilon$$

where the polynomials $M_i''(t', x', y', z')$ are constructed by the $a_i^0(\tau t' + \alpha x' + \beta y')^{n+1}$ in the same way as the polynomials $M_i'(t', x', y', z')$ were constructed by the $M_i^{(n+1)}(t', x', y')$. The polynomials $M_i''(t', x', y', z')$ make the left-hand sides of the equations (1′), respectively (2′), vanish at the point $M^0$, as well as all their partial derivatives by $t', x', y', z'$ up to order $n-1$; respectively, the partial derivatives by $t', x', y'$ up to order $n$.

Suppose that for every $\varepsilon > 0$ the functions $u_i''$ possess in the neighbourhood of the origin bounded partial derivatives up to order $n$ by $t'', x'', y'', z''$. This will be the case if the $u_i$ have on sufficiently smooth surface inside $C$ only such singularities as they have on the surface of $C$, or else singularities that are regularly damped within $C$. Then, by Arzela's theorem, we may choose such an infinite sequence of valence of $\varepsilon$ converging to zero, that in a bounded domain $G''$ around the origin in the space $(t'', x'', y'', z'')$ the functions $u_i''$ together with all their partial derivatives by $t'', x'', y'', z''$ up to order $(n-1)$ will uniformly

converge to certain limit functions $u_i^*$ and their corresponding derivatives. Since

$$u_i''(t'', x'', y'', 0) =$$
$$= a_i(\varepsilon t'', \varepsilon x'', \varepsilon y'')(\tau t'' + \alpha x'' + \beta y'')^{n+1} \ln|\tau t'' + \alpha x'' + \beta y''| +$$
$$+ b_i(\varepsilon t'', \varepsilon x'', \varepsilon y'')(\tau t'' + \alpha x'' + \beta y'')^n |\tau t'' + \alpha x'' + \beta y''| + o(\varepsilon)$$

these limit functions will preserve constant values on all two-dimensional planes lying in $\Pi_3$ and parallel to $\Pi_2$. Since by assumption all $a_i^0$ and $b_i^0$ are not equal to zero, all $u_i^*$ will not be identically equal to zero on $\Pi_3$. Since we suppose that $n \geqslant 3$, the functions $u_i^*$ will satisfy on one side of $\Pi_3$ the equations (1*), and on $\Pi_3$ they will satisfy the linear homogeneous equations (2*), with constant coefficients equal to the values which the corresponding coefficients in the equations (1) and (2) assume at the point $M^0$; these limit equations will contain only terms with the senior derivatives. Since the equations (2*) may be resolved with respect to $\partial u_i^* / \partial z''$, the $\partial u_i^* \partial z''$ will preserve constant values on all two-dimensional planes lying in $\Pi_3$ and parallel to $\Pi_2$.

Thus, $u_i''$ and $\partial u_i'' / \partial z''$ preserve constant values on all two-dimensional planes lying in $\Pi_3$ and parallel to $\Pi_2$, and inside $G''$ the functions $u_i''$ satisfy a system of equations with constant coefficients, which may be resolved with respect to $\partial^2 u_i'' / \partial z''^2$. Hence by the theorem that states that the problem of Cauchy has a unique solution for initial conditions given on $\Pi_3$, the $u_i^*$ will preserve constant values on all two-dimensional planes parallel to $\Pi_2$, not only on those lying in $\Pi_3$. Therefore, inside $G''$ the quantities $u_i''$ may be considered as functions only of $z''$ and $p = \tau t'' + \alpha x'' + \beta y''$. Hence the system of fundamental elasticity equations for the functions $u_i^*$ is reduced to linear equations with constant coefficient containing only second derivatives by $z''$ and $p$.

Thereby become valid those ordinary arguments which serve in the simplest case to determine the propagation velocity of discontinuity waves on a plane surface of an elastic body, be it Rayleigh waves (damped inside the body), or undamped waves.

4. So far we assumed that $n \geqslant 3$. Consider now the case where $n = 2$, or $n = 1$. Let it be supposed that the coefficients of the equations (1) and (2), the surface of the body under consideration, and the surface $S_2$ passing through the point $M^0$ in the cylinder surface $C$ (*i. e.* the discontinuity surface of the senior derivatives of the functions $u_i$), have all a smoothness of order 5. If the discontinuities are extended inside the cylinder $C$ over a certain surface $S_3$, we will suppose these surfaces to have a smoothness of order 5 also. Let the functions $u_i$ themselves have a smoothness of order 4 outside the discontinuity surface, and let the $u_i$ on the surface of the cylinder $C$ unfolded to a plane have the form (3) for $n = 2$, or $n = 1$, where the $a_i$ and $b_i$ have a smoothness of order 4. Let the $u_i$ within $C$ near $S_3$ have a similar form.

Consider first the case where $n = 2$ and the surface $S_2$ has in the neighbourhood of $M^0$ no tangents parallel to the $Ot$-axis. Construct in the neighbourhood of $M^0$, which we will imagine to have the form of a cylinder with the base on the plane $t = t^0 - \varepsilon$, the functions

$$U_i(t, x, y, z) = \int_{t^0-\varepsilon}^{t} u_i(\tau, x, y, z)\, d\tau + M_i(t, x, y, z), \quad i = 1, 2, 3 \qquad (4)$$

where $\varepsilon$ is a small positive number, and the $M_i$ are polynomials. The so defined functions have, as regards smoothness, the same properties which the functions $u_i$ considered above had for $n = 3$. The integrals on the right-hand sides may be differentiated up to three times by $x$, $y$, $z$ under the integral sign. But if we differentiate the same integrals by $t$, we must after differentiation under the integral sign add $u_i(t^0 - \varepsilon, x, y, z)$. Since for a small sum $(x - x^0)^2 + (y - y^0)^2 + (z - z^0)^2$ the point $(t^0 - \varepsilon, x, y, z)$ is outside the surface $S_2$,

we may consider here all functions $u_i(t^0-\varepsilon, x, y, z)$ as possessing continuous derivatives of order 5 with respect to $x$, $y$, $z$. In the same way we must proceed in determining the second derivatives by $t$ of integrals entering into the equations (4). Since the functions $u_i$ satisfy the equations (1) and (2), it is easily seen that after substituting in the left-hand sides of the equations (1) and (2) instead of the $u_i$ the integrals entering into the equalities (4), all terms containing $u_i$ and their derivatives under the integral sign are eliminated, and we obtain a linear combination of coefficients of these equations and of functions $u_i(t, x, y, z)$ and their first derivatives for $t=t^0-\varepsilon$; these linear combinations will have a smoothness of order 4. Since the equations (1), respectively (2), can be resolved with respect to $\partial^2 u_i/\partial z^2$, respectively $\partial u_i/\partial z$, we may choose the polynomials $M_i(t, x, y, z)$ entering into right-hand sides of (4) in such a way that the results of substitution of the functions $U_i(t, x, y, z)$ into the left-hand sides of (1) and (2) will vanish together with all their partial derivatives up to order 3, respectively 4, at the point $M^0$. To the so defined functions $U_i(t, x, y, z)$ all arguments may be therefore applied, which we used above in the case $n \geqslant 3$.

If the surface $S_2$ possessed tangents parallel to the $Ot$-axis, it would have no tangents parallel to one of the axes $OX$, $OY$, say $OX$. Then the functions $U_i$ would have to be constructed by integrating the $u_i$ with respect to $x'$.

For $n=1$ we must take for the construction of the functions $U_i$, instead of ordinary integrals of $u_i$ with respect to $t$, the repeated integrals

$$\int_{t^0-\varepsilon}^{t} dt_1 \int_{t^0-\varepsilon}^{t_1} u_i(t_2, x, y, z)\, dt_2 \quad \text{or} \quad \int_{x^0-\varepsilon}^{x} dx_1' \int_{x^0-\varepsilon}^{x_1} u_i(t', x_2', y', z')\, dx_2'$$

5. The method exposed above may be applied also in the case where the functions $u_i$ have on $\Pi_3$ singularities of some different kind, for instance of the kind

$$(\tau t' + \alpha x' + \beta y')^{n+\alpha}, \quad 0<\alpha<1,$$
$$(\tau t' + \alpha x' + \beta y')^{n+\alpha} \operatorname{sign}(\tau t' + \alpha x' + \beta y'), \quad 0<\alpha<1, \text{ etc.}$$

Let us also observe that in our considerations we used only the following properties of the equations (1) and (2): a) their linearity and homogeneousness, b) the possibility to resolve the equations (1) and (2) with respest to $\partial^2 u_i/\partial z^2$ and $\partial u_i/\partial z$, correspondingly. Our method is not, therefore, confined to the determination of propagation velocity of discontinuities on the surface of an elastic body, but can also be applied to other similar problems.

6. We arrive thus at the following conclusion:

*On the assumptions stated, the propagation velocity of discontinuities of the derivatives of the functions $u_i$ on the surface of an unhomogeneous and, generally speaking, anisotropic elastic body of arbitrary form, free from action of external forces at every point $M$ of its surface, is equal to the propagation velocity of discontinuities of these derivatives on the surface of an infinite homogeneous elastic body bounded by a plane, whose parameters determining its elastic properties are equal to the values of these parameters at the point $M$ of the originally considered body.*

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Received
30.I.1945.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*MATHEMATICAL PHYSICS*

# ELECTRIC CIRCUITS FOR APPROXIMATE SOLUTION OF SYSTEM OF EQUATIONS

By L. I. GUTENMACHER

*(Communicated by N. G. Brujewicz, Member of the Academy, 16.XII.1944)*

The principal elements of the circuits proposed are four poles consisting of Γ-, T-, Π- and X-shaped connexion schemes of bipoles formed by arbitrary combinations of active ($R$) and reactive ($L$, $M$ and $C$) elements of the electric circuit.

The transien conductance of such a bipole is determined by the expression

$$A,\ B = 1 : \sum_{i=1}^{n} \left[ 1 : \sum_{m=1}^{k_i} \frac{1}{R_{im} + I_{im}p + \frac{r}{C_{im}p}} \right] \tag{1}$$

where $R_{im}$ is resistance, $I_{im}$ is inductance, $C_{im}$ is capacity, and $p = \partial/\partial t$.

For the alternating current with frequency $\omega$

$$A,\ B = 1 : \sum_{i=1}^{n} \left[ 1 : \sum_{m=1}^{k_i} \frac{1}{R_{im} + i\left(\omega L_{im} + \frac{1}{\omega C_{im}}\right)} \right]$$

Four-poles represent a system possessing two terminals (the poles 1a, 1b) for the input of energy and two (2a, 2b) for the output. By the transmission coefficient $N_{12}$ of a four-pole we shall mean the ratio of the potential difference $u_2$ or the current strength $I_2$ in the outlet circuit (2a, 2b) to the voltage $u_1$ or current strength $I_1$ in the inlet circuit (1a, 1b).

In a directional four-pole, or amplifier, we shall agree to put $N_{12} = \text{const} = N$, and $N_{21} = 0$.

Consider a series of $n$ four-poles (Figs. 1 and 2). In one case (Fig. 1) they are linked up by current coupling; in the other case (Fig. 2), by voltage coupling.

We write now the equations for current strengths $I_l$ at the junction points $l$ in the inlet circuits of the four-poles $N$ (Fig. 1):

$$I_1 = A_{11}[a_{11}(N_1u_1 + E_1) - u_1] + A_{12}[a_{12}(N_2u_2 + E_2) - u_1] + \ldots + A_{1n}[a_{1n}(N_nu_n + E_n) - u_1]$$

. . . . . .

. . . . . .

$$I_n = A_{n1}[a_{n1}(N_1u_1 + E_1) - u_n] + A_{n2}[a_{n2}(N_2u_2 + E_2) - u_n] + \ldots + A_{nn}[a_{nn}(N_nu_n + E_n) - u_n]$$

or

$$I_l = \sum_{s=1}^{n} A_{ls} a_{ls} E_s - \sum_{s=1}^{n} A_{ls} u_l + \sum_{s=1}^{n} A_{ls} a_{ls} N_s \dot{u}_s \tag{2}$$

Since the relation between the input voltage $u_l$ and the current strength $I_l$

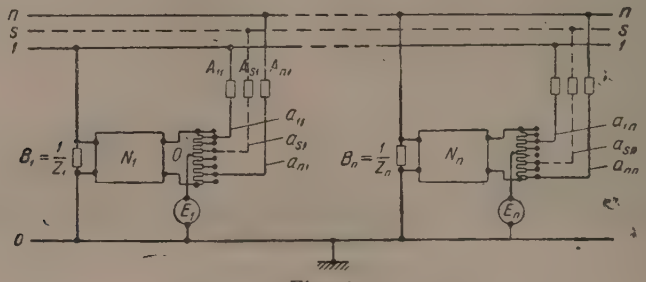


Fig. 1

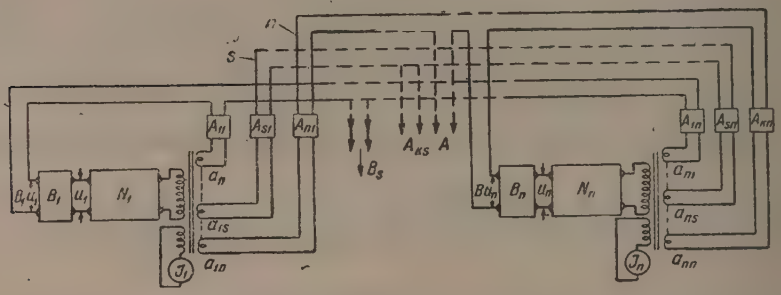


Fig. 2

is determined by Ohm's law

$$u_l = Z_l I_l \quad \text{or} \quad I_l = \frac{U_l}{Z_l} = B_l u_l$$

the equation (2) may be rewritten in the form

$$I_l = F_l - \sum_{s=1}^{n} A_{ls} Z_l I_l + \sum_{s=1}^{n} A_{ls} a_{ls} N_s Z_s I_s \tag{3}$$

or in the form

$$B_l u_l = F_l - \sum_{s=1}^{n} A_{ls} u_l + \sum_{s=1}^{n} A_{ls} a_{ls} N_s u_s \tag{4}$$

where

$$F_l = \sum_{s=1}^{n} A_{ls} a_{ls} E_l \quad (l = 1, 2, \ldots, n)$$

The equation (3) is formed for current strengths, and the equation (4) for voltages. From these fundamental equations we may obtain equations for the charges $Q$ ($Q = \int i dt$) and for the power and energy of the system.

In these equations $A_{ls}$ denote coupling conductances (Fig. 1); $a_{ls}$ are constant coefficients corresponding to the portions of the voltage applied to the potential dividers inserted in the outlet circuits $N$; $E$ are the e. m. f. of extraneous current sources; $N_s$ are transmission coefficients (or amplification coefficients). From these equations we may obtain for different working conditions of the system many important equations of mathematical physics.

In establishing the voltage coupling between the four-poles $N$ (Fig. 2), it is necessary to isolate their circuits in order to obtain a summation of voltages. To this end it is possible to apply insulated e. m. f. sources or mutual inductances (transformers).

In contradistinction to the scheme in Fig. 2, the elements $A$ and $B$ are no bipoles, but four-poles with transmission coefficients $A$ and $B$. In this particular case a multiwinding transformer is inserted at the output of every amplifier. To one of the windings the current source $I_{os}$ is connected. The secondary circuits $A_{ls}$ and $B_l$ are closed in $n$ coils. The sum of voltages in each coil is equal to zero. We now can write the equations of the whole system as follows:

$$B_1 u_1 = A_{11} a_{11} p (I_1 + I_{01}) + A_{12} a_{12} p (I_2 + I_{02}) + \ldots + A_{1n} a_{1n} p (I_n + I_{0n})$$

. . . . . .

. . . . . .

$$B_n u_n = A_{n1} a_{n1} p (I_1 + I_{01}) + A_{n2} a_{n2} p (I_2 + I_{02}) + \ldots + A_{nn} a_{nn} p (I_n + I_{0n})$$

or

$$B_l u_l = \sum_{s=1}^{n} A_{ls} a_{ls} p I_s + \sum_{s=1}^{n} A_{ls} a_{ls} p I_{0s} \tag{5}$$

Since the current strengths $I_k$ are functions of the inlet voltages of the four-poles $N_{ko}$

$$I_k = N_k u_k$$

after substitution:

$$B_l u_l = F_l + \sum_{s=1}^{n} A_{ls} a_{ls} N_s p u_s \tag{6}$$

where

$$F_l = \sum_{s=1}^{n} A_{ls} a_{ls} p I_{0s} \quad (l = 1, 2, \ldots, n)$$

The proposed method of building up artificial circuits can also be used to construct mechanical, electro-mechanical and hydraulic circuits from four-poles (amplifiers) and bipoles consisting of concentrated parameters (springs, masses, dampers, containers, servo-mechanisms, *etc.*).

Such circuits are practically useful for approximate solutions of systems of differential equations (transient processes in the scheme) and of algebraic equations (stable processes).

In solving differential equations the initial conditions must be determined by the voltages applied to condensers and by the currents in the inductances at the initial instant.

The described systems of electric circuits with amplifiers and the corresponding mechanical and electro-mechanical systems are a generalization and further development of well known mechanical and electrical devices for the solution of systems of equations ([1–4]). The use of amplifiers greatly widens their possibilities.

Later, the author will give an account of the practical results obtained in developing the electro-mechanical methods of solution of systems of equations along the lines indicated in this paper.

Received
16. XII. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*PHYSIQUE APPLIQUÉE*

# AUTO-OSCILLATIONS D'UN SCHÉMA SIMPLIFIÉ CONTENANT UNE HÉLICE A PAS VARIABLE AUTOMATIQUE

Par A. ANDRONOW, N. BAUTIN et G. GORELIK

*(Présenté par L. I. Mandelstam, de l'Académie, le 22.XI.1944)*

Le travail de V. Bodner (1) a attiré notre attention sur l'étude des auto-oscillations qui accompagnent parfois le fonctionnement des hélices à pas variable automatique. Nous examinons ici un schéma plus simpe que celui de Bodner, faisant abstraction de la variation de vitesse de l'avion et de l'inertie de l'indicateur; en revanche nous donnons une solution exacte, permettant d'établir un tableau complet de l'influence des divers paramètres sur l'existence des auto-oscillations, et des formules exactes pour leur amplitude, leur période et les conditions de leur anéantissement. Nous considérons l'ensemble vilebrequin — pales de l'hélice — indicateur comme un système à trois degrés de liberté seulement, dont la configuration est donnée par les angles de pivotement $\theta$ et $\varphi$ du vilebrequin et des pales et par le déplacement $\zeta$ du manchon de l'indicateur. Nous le considérons de plus comme un système dégénéré: d'abord parce que nous négligeons l'inertie de l'indicateur et du servomoteur, et ensuite parce que nous admettons que le frottement de l'indicateur obéit à la loi de Coulomb*. Le problème considéré, proche du problème classique de Léauté (2), répond à plusieurs autres dispositifs de régulation automatique.

1. Ecrivons les équations du mouvement du vilebrequin

$$J\frac{d\omega}{dt} = M_1(\omega, \lambda) - M_2(\omega, \varphi) \qquad (1a)$$

($\omega = \dot{\varphi}$, $t$ est le temps, $J$ le moment d'inertie, $\lambda$ un paramètre caractérisant l'admission des gaz, $M_1$ le moment actif du moteur, $M_2$ le moment des résistances de l'hélice), du manchon

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) - \frac{\partial T}{\partial \xi} = -\frac{dV}{d\xi} + Q(\dot{\zeta}) \qquad (1b)$$

($T = T(\zeta, \dot{\zeta}, \omega)$ est l'énergie cinétique du système, $V(\zeta)$ l'énergie potentielle des ressorts et des poids de l'indicateur, $Q(\dot{\zeta})$ la force de frottement dans

* Il est supposé que dans notre schéma la variation de pas doit permettre de faire varier la traction par réglage de l'admission des gaz, sans faire varier la vitesse angulaire du moteur.

l'indicateur) et du servomoteur

$$J_0 \frac{d^2\varphi}{dt^2} = P_1(\zeta) - P_2(\dot{\varphi}, \varphi) \tag{1c}$$

($J_0$ est le moment d'inertie, $P_1(\zeta)$ le moment des forces actives, $P_2(\varphi, \varphi)$ celui des forces de résistance).

Considérons le voisinage du régime d'équilibre $\omega_0$, $\zeta_0$, $\varphi_0$. Effectuant le changement de variables $\omega = \omega_0 + \eta$, $\zeta = \zeta_0 + \zeta$, $\varphi = \varphi_0 + \chi$, nous obtenons après des simplifications évidentes et négligeant l'inertie de l'indicateur et du servomoteur, les équations du mouvement (1):

$$\frac{d\eta}{dt} = -M\eta - N\chi,\ a\xi - b\eta + Q(\dot{\zeta}) = 0,\ \frac{d\chi}{dt} = F(\zeta) \tag{2}$$

Les constantes $M$, $N$, $a$, $b$ sont positives. Nous admettons (loi de Coulomb)

$$Q(\dot{\xi}) = -K \text{ pour } \dot{\zeta} > 0,\ Q(\dot{\xi}) = +K \text{ pour } \dot{\xi} < 0,\ |Q(\dot{\xi})| < K \text{ pour } \xi = 0$$

$K$ étant une constante positive. La 2-ème équation (2) donne alors:

$$a\xi - b\eta = \begin{cases} -K \left(\text{d'où } \dot{\xi} = \frac{b}{a}\dot{\eta}\right) \text{ pour } \dot{\xi} > 0 \\ +K \left(\text{d'où } \dot{\xi} = \frac{b}{a}\dot{\eta}\right) \text{ pour } \dot{\xi} < 0 \end{cases} \quad |a\xi - b\eta| < K \tag{3}$$

Nous représenterons la caractéristique de servomoteur par la ligne brisée usuelle

$$F(\xi) = -F(-\xi) = \begin{cases} 0 & \text{pour } |\xi| < \bar{\psi}_0 \\ \bar{\alpha}(\xi - \bar{\psi}_0) & \text{pour } \bar{\psi}_0 \leq \xi < \bar{\psi}_1 \\ +\bar{A} & \text{pour } \xi > \bar{\psi}_1 \end{cases}$$

Posons

$$\left.\begin{array}{l} \frac{a\xi}{2K} = x;\ \frac{b\eta}{2K} = y;\ -\frac{b}{2K}\left(\eta + \frac{N}{M}\chi\right) = z;\ Mt = \tau' \\ \frac{a\bar{\psi}}{2K} = \psi_0;\ \frac{a\bar{\psi}_1}{2K} = \psi_1;\ \frac{Nb}{2KM^2}\bar{A} = A;\ \left(\alpha = \frac{A}{\psi_1 - \psi_0}\right) \end{array}\right\} \tag{4}$$

et interprétons $x$, $y$, $z$ comme des coordonnées cartésiennes. Le système (2) prend l'aspect

$$\frac{dx}{d\tau'} = f(x, y, z),\ \frac{dy}{d\tau'} = z,\ \frac{dz}{d\tau'} + z = g(x) \tag{5}$$

En vertu de (3) le point figuratif se trouve soit entre les plans $y - x = \pm 1/2$, soit sur le demi-plan $(H)$: $x - y = 1/2$, $z < 0$, soit sur le demi-plan $(H')$: $x - y = -1/2$, $z > 0$; $f \equiv 0$ entre les plans $x - y = \pm 1/2$; $f \equiv z$ sur les demi-plans $H$ et $H'$; $g(x) = -g(-x)$ où $g(x) \equiv -A$ pour $x > \psi_1$; $g(x) \equiv -\alpha(x - \psi_0)$ pour $\psi_0 < x < \psi_1$; $g(x) \equiv 0$ pour $|x| < \psi_0$*.

2. En vertu de (5): 1) la partie intéressante (entre $H$ et $H'$) de l'extension en phase à trois dimensions est remplie de morceaux de trajectoires parallèles au plan $x = 0$; 2) la division en trajectoires de $H$ et $H'$ comprend des segments de droites (pour $|x| < \psi_0$), des morceaux de trajectoires appartenant à un foyer

* Le problème se réduit pour $\psi_0 = 0$, $A = \infty$ à celui déjà étudié par Andronow et Bautin (3).

ou un noeud stable pour $|\psi_0| < |x| < |\psi_1|$, des morceaux des courbes $x = A - z - A \ln C_1(z - A)$ pour $x < -\psi_1$, et $x = -A - z - A \ln C(z + A)$ pour $x > \psi_1$; la jointure des morceaux de trajectoires s'opère sur les droites $x = \pm\psi_0$ et $x = \pm\psi_1$; 3) la bande $|x| < \psi_0$, $|x - y| < {}^1/_2$, $z = 0$ est une bande d'équilibre. Le point figuratif, lorsqu'il se meut sur $H$ ou $H'$, rejoint soit un des bords de la bande d'équilibre soit une des demi-droites $(L)$: $x - y = {}^1/_2$, $z = 0$, $x < -\psi_0$, ou $(L)$: $x - y = -{}^1/_2$, $z = 0$, $x > \psi_0$; dans ce dernier cas il quitte $H$ ou $H'$ à destination de $H'$ ou $H$ (fig. 1). L'extension en phase est symétrique par rapport à l'origine de coordonnées.

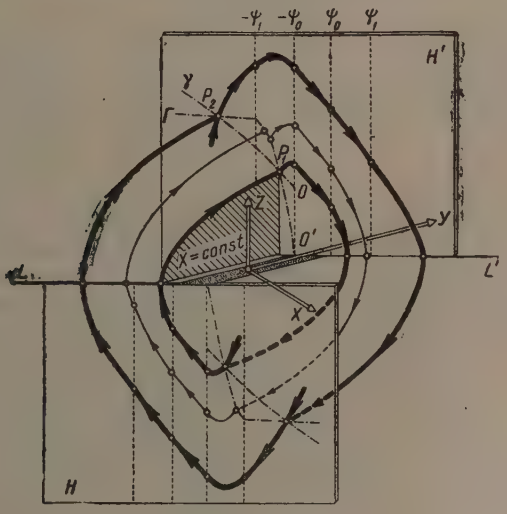


Fig. 1

Appelons transformation $S^+$ (ou $S^-$) le transfert d'un point de $L$ (ou $L'$) dans le demi-plan $H'$ (ou $H$), et transformation $E^+$ (ou $E^-$) le transfert d'un point de $H'$ (ou $H$) sur la demi-droite $L'$ (ou $L$).

3. La transformation $S^+$ trasforme $L$ en une courbe $\Gamma$:

$$\left.\begin{aligned} &x + \psi_0 = \frac{1}{\alpha(1 - \tau - e^{-\tau})}, \quad z = \frac{1 - e^{-\tau}}{e^{-\tau} + \tau - 1}, \quad 0 < \tau < \infty, -\psi_1 \leqslant x \leqslant -\psi_0 \\ &z = A(1 - e^{-\tau^*}), \quad A(e^{-\tau^*} + \tau^*) = 1 + A, \quad x \leqslant -\psi_1 \end{aligned}\right\} \quad (\Gamma)$$

Le lieu des points tels que $|x|$ ne varie pas en résultat de $E^+$ est une courbe $\gamma$ déterminée pour $x \leqslant -\psi_1$ par l'equation

$$\sqrt{u^2 - u(\psi_1 - \psi_0) + \alpha(\psi_1 - \psi_0)^2}\, \exp\left[-\frac{1}{\sqrt{4\alpha - 1}} \operatorname{arctg} \frac{(\psi_1 - \psi_0)\sqrt{4\alpha - 1}}{2u - (\psi_1 - \psi_0)}\right] -$$

$$-\sqrt{v^2 + v(\psi_1 - \psi_0) + \alpha(\psi_1 - \psi_0)^2}\, \exp\left[\frac{1}{\sqrt{4\alpha - 1}} \operatorname{arctg} \frac{(\psi_1 - \psi_0)\sqrt{4\alpha - 1}}{2v + (\psi_1 - \psi_0)}\right] = 2\psi_0$$

$$(0 < \operatorname{arctg} < \pi)$$

où

$$A - u = (A - z)\exp\left(\frac{\psi_1 + x + z - u}{A}\right), \quad v + A = \exp\left(\frac{\psi_1 + x + v}{A}\right),$$

et pour $-\psi_1 \leqslant x \leqslant -\psi_0$ par l'équation

$$\sqrt{z^2+z(x+\psi_0)+\alpha(x+\psi_0)^2}\ \exp\left[-\frac{1}{\sqrt{4\alpha-1}}\ \operatorname{arctg}\ \frac{-(x+\psi_0)\sqrt{4\alpha-1}}{2z+x+\psi_0}\right]+$$

$$+(x+\psi_0)\sqrt{\alpha}\ \exp\left[\frac{1}{\sqrt{4\alpha-1}}\ \operatorname{arctg}\ \sqrt{4\alpha-1}\right]=2\psi_0$$

$$(0<\operatorname{arctg}<\pi)$$

Les points de $H$ sont divisés par $\gamma$ en 3 classes: a) points appartenant à $\gamma$; $E^+$ n'en fait pas varier $|x|$; b) points situés entre $\gamma$ et le prolongement de $L'$; $E^+$ en diminue $|x|$; c) tous les autres points; $E^+$ en augmente $|x|$.

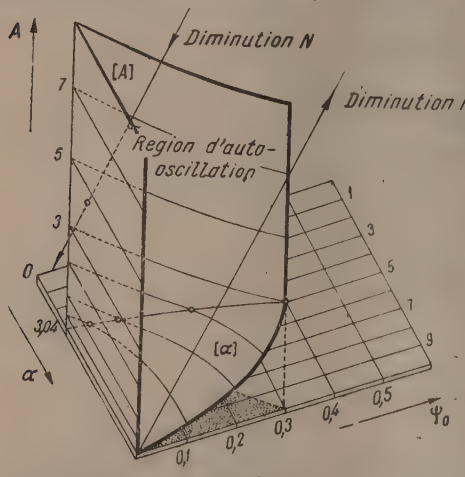

Fig. 2

4. Les courbes $\Gamma$ et $\gamma$ se coupent en deux points différents, ou bien se coupent en deux points confondus, ou bien ne se coupent pas. Soit $P_1$ la première intersection (à partir de $x = -\psi_0$), $P_2$ la seconde. Par $P_1$ passe alors un cycle limite instable, par $P_2$ un cycle limite stable. Si la perturbation initiale* est suffisamment petite, la système tend vers un état d'équilibre; dans le cas contraire s'établissent des auto-oscillations. Si $\gamma$ et $\Gamma$ ne se coupent pas, l'équilibre s'établit quelles que soient les conditions initiales.

5. Nous avons ramené les paramètres de notre problème dynamique à 3 paramètres essentiels $\psi_0$, $\alpha$, $A$. La fig. 2 représente le partage de l'espace des paramètres en regions différant par la structure qualitative de l'extension en phase. La région où des auto-oscillations sont possibles, est comprise entre le plan $\psi_0 = 0$, la surface cylindrique $[A]$ à génératrices parallèles à l'axe $A$, et la surface $[\alpha]$ contenant l'axe $\alpha$.

La fig. 2 montre une partie de cette région limitée par les plans arbitraires $\alpha = \text{const}$, $A = \text{const}$. En dehors de cette région le système tend vers un état d'équilibre stable**.

La fig. 2 et les formules (4) indiquent que l'on peut toujours se débarrasser des auto-oscillations, par example, par l'un quelconque des procédés les plus

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* Due, par example, à une variation brusque de $\lambda$ survenue à l'époque $\tau' = 0$; si l'état du système pour $\tau' = 0$ était d'équilibre, il ne l'est plus pour la nouvelle valeur de $\lambda$.

** Il ressort de la fig. 2 que: 1) lorsque $\alpha < 3{,}04$ il ne peut y avoir d'auto-oscillations; 2) à tout $\alpha > 3{,}04$ correspond une valeur $\psi_0 = \psi_0'$ (elle croît avec $\alpha$) telle que lorsque $\psi_0 > \psi_0'$ il ne peut y avoir d'auto-oscillations: 3) à tous $\alpha > 3.04$; $\psi_0 < \psi_0'$ correspond une valeur $A = A'$ (elle croît avec $\psi_0'$) telle que pour $A > A'$ les auto-oscillations sont possibles, et pour $A < A'$ impossibles 4) à tout $A$ correspond une valeur $\psi_0 = \psi_0^*$ telle que lorsque $\psi_0 < \psi_0^*$ il ne peut y avoir d'auto-oscillations.

simples ci-dessous: 1) diminution du frottement $K$; 2) diminution de $\alpha$; 3) diminution de $\overline{A}$; 4) élargissement de la zone d'insensibilité $\psi_0$; 5) diminution du pouvoir régulateur $N$ de l'hélice.

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Manuscrit reçu
le 22. XI. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*TECHNICAL PHYSICS*

# MECHANISM OF DIELECTRIC BREAKDOWN OF LIQUIDS

By V. S. KOMELKOV

*(Communicated by K. I. Shenfer, Member of the Academy, 16.XI.1944)*

In this work are described the results of a study of the breakdown mechanism for two liquids of very different molecular structure—the transformer oil and the water. The former is endowed with but feebly expressed polar properties, whereas the latter is a distinctly polar liquid. The transformer oil was technically pure, and the water was distilled water with a residual conductivity of $10^{-5}$ ohm$^{-1}$ cm$^{-1}$.

The equipment and method used in the experiment have been described by the author in an earlier paper ([1]) which dealt with a discharge in the air under normal conditions. This time instead of a cable a screened overhead line was used. It was two metres long and connected a cathod oscillograph to a resistance $R_{\text{mes.}}$ inserted into the circuit of a grounded electrode. The liquid was in a glass container having the form of a rectangular prism, $20 \times 25 \times 45$ cm. Through the bakelite lid of the container passed a porcelain insulator with a copper core, which served as a high-voltage lead-in. The lower, pointed end of the copper core was submerged in a liquid and presented a needle electrode. The grounded electrode, made likewise of copper, passed through an opening in the bottom of the container.

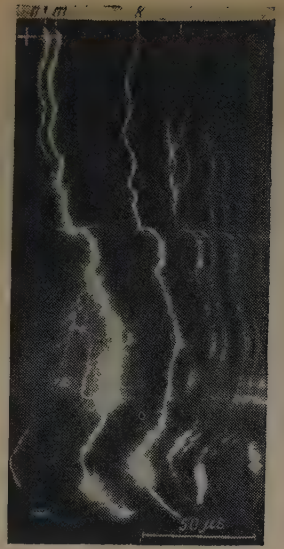
The discharges took place between two types of spark-gap: needle—needle and needle—plane. The maximum impulse strength of both positive and negative polarity amounted to 500 kV. The experiments were run at a minimum impulse voltage. As is known, the insertion into the discharge circuit of a resistance ($R_1$) inhibiting the development of the spark permitted I. S. Stekolnikov and A. P. Beliakov to slow down the spark in the air, and thus to measure the rate of the ionization processes by optical means ([2]).

By inserting the resistances $10^4$, $10^5$, $10^6$ ohm, the present author was able to slow down the discharge effectively in liquid also. Thereby he obtained upon a moving film and oscillograms, records by means of which the mechanism of the development of breakdown could be established. The following three main stages were observed by him in the puncture of long intervals under conditions of an unhomogeneous field:

1) Preliminary breakdown, when the ionization process in a virgin medium runs with a finite velocity from electrode to electrode, and closes the latter together by a conducting path. To draw an analogy with the discharge in gases, this stage can be described as l e a d i n g stage.

2) B r i g h t f l a s h, which appears after the close of the leading stage; this stage differs from the leading one in that the channel of the spark emits a light of a many times greater brightness and of a very short duration.

3) A r c d i s c h a r g e. The leading charge starts from the electrode having the greater curvature. In the case of two pointed-end electrodes, there are two leaders moving towards each other.



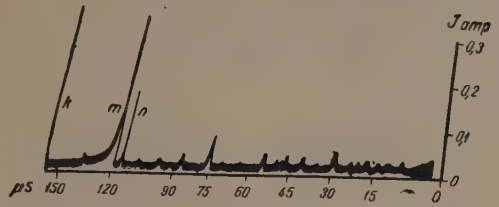



Fig. 1. Photograph of a discharge in oil with a $10^6$ ohm inhibition Two pointed-end electrodes, 12 cm apart. Below, oscillogram of the same discharge

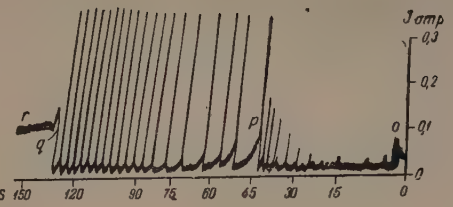
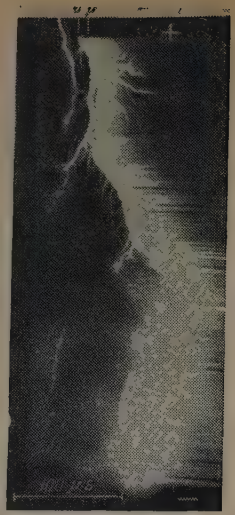



Fig. 2. Photograph of a discharge in oil with a $1.4 \cdot 10^6$ ohm inhibition. Two pointed-end electrodes, 12 cm apart. Below, oscillogram of the same discharge



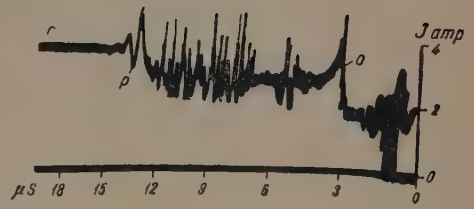



Fig. 3. Photograph of a discharge in distilled water with $10^5$ ohm inhibition. Two pointed-end electrodes, 9 cm apart. Below, oscillogram of the same discharge

The leader in the oil emits a characteristic greenish-yellow light which is usually so faint as to be difficult to record on a film. Contrary to this, the leader in water emits an intense light of bluish-violet colour (its intensity is much higher than that of the light due to a discharge in the air). The difference in the intensity of the visible light may be accounted for by the different absorbing power of the liquids and by the different structure of their molecules. A light of marked intensity is only observed at the head of the leader; the trajectory passed by the latter emits a much fainter light.

In the initial phase the velocity of the leader is a maximum. As the leader moves farther into the spark-gap, it loses speed and sometimes stops altogether. A while after stopage (negative leader) or when the movement is still in progress (positive and negative leader), there occurs a break-down of the leader's channel, accompanied by an intense light emission from the whole trajectory travelled by the discharge. Afterwards the movement regains its maximum velocity only to be slowed down again later, and so on. It will be obvious that the ionization process develops at a variable speed, continuous motion being combined with stepwise motion.

The branches formed by the discharge have always the same direction with the movement of the leader. To every step of the leader there corresponds an impulse of the current having a maximum at the breakdown of the channel and a minimum at the greatest reduction of the leader's velocity, or at its stoppage.

Figs. 1 and 3 are photographs and oscillograms of the same discharges, taking place in the oil and water, respectively. In Fig. 1 only the last step of the leader *n* can be seen, and the oscillogram shows the corresponding impulse of the current *n*. The preceding steps whose presence and number can be gathered from the oscillogram, are not recorded by the photograph, because their intensity of light emission was too small. The time intervals between the separate steps of the leader are larger and the number of steps is smaller in the oil than in water. In the case of water, the course of the current oscillogram is complicated by the presence of the component due to residual conductivity. The initial value of this component can be determined directly from the oscillogram.

The adjoined table gives the main characteristics of the discharges.

| Liquid | Resistance of inhibition in ohm | Electrodes | Maximum current of leader, in A | | Mean current of leader, in A | | Current of arc, in A | | Distance between electrodes, in cm | | Duration of leading stage, in microsec. | | Mean velocity of leaders, in cm/sec | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | + | − | + | − | + | − | + | − | + | − | + | − |
| Transformer oil | $10^5$ | pointed end | | | | | | | | | | | | |
| | | plane | 1.0 | 1.1 | 0.3 | 0.35 | 1.4 | 2.0 | 18.5 | 15.5 | 100 | 21 | $1.4 \cdot 10^5$ | $5.7 \cdot 10^5$ |
| | $10^6$ | pointed end | | | | | | | | | | | | |
| | | plane | 0.42 | 0.4 | 0.1 | 0.1 | 0.11 | 0.11 | 16.5 | 14.0 | 90 | 175 | $1.4 \cdot 10^5$ | $6.1 \cdot 10^4$ |
| Distilled water | $10^4$ | pointed end | | | | | | | | | | | | |
| | | pointed end | — | | — | | — | | 20 | | 9.1 | | $2.64 \cdot 10^6$ | — |
| | $10^5$ | do | 2.8 | | 1.32 | | 3.4 | | 11.7 | | 9.5 | | $1.67 \cdot 10^6$ | — |

It has already been pointed out that the velocity of the leader changes from point to point. The table gives the mean velocities which may be several times smaller than the maximum velocity. With further decrease in the inhibition (below $10^5$) the velocities of the leaders (negative leaders, especially) attain still higher values, exceeding the probable velocity of electrons in a liquid medium. Just as in the case of gases, this effect can be accounted for by the photo-ionization of the molecules of the liquid caused by the radiation of the developing discharge.

The velocities of negative leaders upon the breakdown of water are many times less than the velocities of positive leaders, and sometimes defy the attempts

to measure them. While in the case of oil the length of the positive leader upon a $10^5$ ohm inhibition between two pointed-end electrodes is 1.2—1.5 times larger than the length of the negative leader, this difference is as great as 12—15 times in the case of water under similar conditions.

The leading stage is separated from the arc stage by a sharp flash which extends over the whole channel and is more intense than the light emitted by the leader. Of maximum brightness is the point where the two leaders meet; away from it in the direction of the electrode the light gradually weakens. It is likely that the light propagates along the channel in the same directions with a very high velocity. The current maxima in the flash stage exceed the currents of the leader and arc several times. Their appearance can easily be explained, if one takes into account that the potential of the channel decreases after the leaders have met together, and there occurs a reverse flow of the charges concentrated on the periphery of the channel and on the neighbouring particles of the liquid. Moreover. in the flash stage a discharge of electrode capacity sets in, which adds its current component to the arc current. The appearance of such current impulses (in the main discharge) when the spark process takes place in the atmosphere was observed by the author before and is described in detail in [1].

From $R + 120\,000$ ohm onwards a very remarkable phenomenon is observed in the oil. After the leading stage is closed, the ark does not appear at once, but rather after a series of breakdowns which increase in number with the growth of inhibition. Fig. 2 presents the optical picture and record of the current for $R_i = 10^6$ ohm.

During the interval $o-p$ lasts the leader stage, then follow the repeated impulses $p-q$, the arc $q-r$, and finally, the impulses again $r-s$. A very curious phenomenon is the breaking of the arc and the resumption of reiterating charges in the closing phase where the voltage of the circuit and the currents in the discharge become small enough. The intervals between the repeated impulses reach a value of 7—12 microseconds; they decrease markedly with the increase of the ordinal number of the impulse. In first approximation the repeated discharges can be explained by the self-restoration of the dielectric strength of the spark-gap at the expense of recombination processes and through capture of the charge-carriers by neutral particles.

The dielectric breakdown of liquids as described above cannot be interpreted satisfactorily on the basis of the existent theories which ignore the part played by the electron avalanche process. From the striking similarity between the separate discharge phases in liquids and in gases one is permitted to suppose that in liquids, in spite of considerable differences in the conditions of electron movement (high probability of stepwise ionization, energy loss by the excitation of elastic atomic vibrations in the molecule, adhesion of electrons, *etc.*), effects favouring the formation of avalanches arise upon breakdown. These effects may consist in the photo-ionization (or photo-excitation) due to the radiation of the discharge itself, and the decrease in the work of ionization under the action of the electric field on the molecules of the liquid, which propagates along with the leader.

The step-like nature of the leader process is probably due not so much to the ionization mechanism as to the ageing of the channel; the conductivity of the latter decreasing with time, there eventually occurs a drop of gradient in the region of ionization, whereby the movement of the charge-carriers is slowed down.

The author expresses his gratitude to I. S. Stekolnikov for the attention paid to this work.

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Received
16. XI. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*PHYSICAL CHEMISTRY*

# DETERMINATION OF THE FLOW VELOCITY OF THE DETONATION PRODUCTS OF CONDENSED EXPLOSIVES

By L. D. LANDAU and K. P. STANIUKOVICH

*(Communicated by P. L. Kapitza, Member of the Academy, 24.VII.1944)*

In determining the flow velocity of detonation products it was established that in certain cases this velocity exceeds the detonation velocity. An especially high flow velocity was observed by G. I. Pokrovsky (1) and R. Becker (2) in experiments with cumulative charges during detonation in a tube. To-day this is a well-known and repeatedly verified phenomenon. Moreover, for experiments approaching the case of one-dimensional detonation, the flow velocities for commonly employed explosives (tol, tetryl) lie within the limits 8000 – 12000 m/sec.

The case of one-dimensional detonation lends itself comparatively easily to theoretical analysis, while the flow velocity $u$ of the detonation products is determined from the Riemann solution of the hydrodynamics equation, which may be written in the following form:

$$u = u_{\text{initial}} + \int_{\text{initial}}^{v} v_{\text{initial}} \sqrt{-dp\,dv} \qquad (1)$$

where $p$ is the pressure; $v$ is the specific volume; $u_{\text{initial}}$ is the velocity of the detonation products; $v_{\text{initial}}$ is their specific volume at the front of the detonation wave.

As we have shown elsewhere (3), the following law connecting $p$ and $v$ (which replaces the usual adiabatic law for an ideal gas) holds for the case of detonation:

$$pv^n = \text{const} \qquad (2)$$

where $n = 3$ for standard local action explosives.

Hence, the following formulae will hold:

$$u_{\text{initial}} = \frac{D}{n+1} \qquad (3)$$

$$c_{\text{initial}} = \frac{nD}{n+1} \qquad (3a)$$

$$v_{\text{initial}} = v_0 \frac{n}{n+1} \qquad (4)$$

where $D$ is the detonation velocity; $c_{\text{initial}}$ is the velocity of sound at the front of the detonation wave, and $v_0$ is the initial volume of the explosive.

Let us evaluate the integral $\int v_{\text{initial}} \sqrt{-dp\,dv}$. To do this we will divide the domain of integration in two. Moreover, at the point $(p_k, v_k)$ we will conjugate the solution obtained for the expansion products that obey equation (2), and the solution obtained for an ideal gas. Then

$$u = u_{\text{initial}} + \frac{2c_{\text{initial}}}{n-1} - \frac{2c_k}{n-1} + \int_{T}^{T_k} \sqrt{\frac{c_p c_v}{RT}}\, dT \tag{5}$$

where $T_k$ is the temperature at the point of conjugation; $c_v$ and $c_p$ are specific heat at constant volume and constant pressure; $R$ is specific gas constant. This temperature may be obtained from the equation

$$\Delta Q = \int_0^{T_k} c_v dt$$

where $\Delta Q$ is the residual energy at the point of conjugation determined by the equations:

$$\frac{p_{\text{initial}} v_{\text{initial}}}{n-1} - \frac{p_k v_k}{n-1} + \Delta Q = E = \frac{p_{\text{initial}} + p_0}{2} (v_0 - v_{\text{initial}}) + Q$$

where $Q$ is the heat of reaction.

From this, making use of formulae (3) and (4) and discarding the quantities $p_k v_k/(n-1)$ and $2c_k/(n-1)$ which are small in comparison with $\Delta Q$, we get

$$\Delta Q = Q - \frac{D^2}{2(n^2-1)}$$

Now for the determination of $u$ we will have the formula:

$$u = \frac{3n-1}{n^2-1} D + \int_0^{T_k} \sqrt{\frac{c_p c_v}{RT}}\, dt \tag{5a}$$

If we disregard the dependence of $c_p$ and $c_v$ upon the temperature, the formulae are considerably simplified, becoming:

$$u = \frac{3n-1}{n^2-1} D + \frac{2}{\gamma-1} \left[ \sqrt{\gamma RT_k} - \sqrt{\gamma RT} \right]$$

($\gamma = c_p/c_v$). Inasmuch as $T_k = \Delta Q/c_v$, the formula for the determination of $u$ will have the following final form:

$$u = \frac{3n-1}{n^2-1} D + 2\sqrt{\frac{\gamma}{\gamma-1}\Delta Q} - \frac{2}{\gamma-1}\sqrt{\gamma RT} \tag{6}$$

As computations show, the value of $T_k$ is comparatively small (of the order of 1000—1500°), and it indeed becomes possible to discard the dependence of the specific heat upon the temperature. Moreover, the error in determining $u$ will not be greater than 5 per cent.

For the case of gas flow into vacuum, $T$ has to be placed equal to zero. For the determination of the flow velocity into the atmosphere $u_x$, it is necessary to take into account the resistance of the shock-wave forming before the front of the moving gas.

With this aim in view let us determine the velocity $D_{\text{shock}}$ and the pressure $p_{\text{shock}}$ at the front of the shock-wave:

$$D_{\text{shock}} = \frac{\gamma_a + 1}{2} u_x \tag{7}$$

$$p_{\text{shock}} = \frac{\gamma_a + 1}{2} \rho_a u_x^2 \tag{8}$$

where $\rho_a$ is the density of the air, and $\gamma_a$ is the ratio of its specific heats. It is obvious that the condition $p_{\text{shock}} = p_x$, where $p_x$ is the residual pressure of the detonation products, makes it possible to determine the velocity sought for. For this it is necessary to know $p_k$ and $v_k$ — the pressure and volume at the point of conjugation.

These quantities are determined from the equations:

$$p_k v_k^n = p_{\text{initial}} v_{\text{initial}}^n \quad \text{and} \quad p_k v_k = (\gamma - 1)\,\Delta Q$$

Since the value of $\gamma_a$ cannot be determined exactly owing to the high temperature at the front of the shock-wave, it may be assumed, without more than slightly affecting the accuracy of the computation, that $\gamma_a = 1$, and expressions (7) and (8) may be rewritten in the form:

$$D_{\text{shock}} = u_x \tag{7a}$$

$$p_{\text{shock}} = \rho_a u_x^2 \tag{8a}$$

Further it is obvious that $\sqrt{\gamma R T_x} = c_k \left(\frac{p_x}{p_k}\right)^{\frac{\gamma-1}{2\gamma}}$, and from this it follows that

$$u = \left(\frac{2}{\gamma+1}\right)^{1/2} \left(\frac{p_x}{\rho_a}\right)^{1/2} = \frac{3n-1}{n^2-1} D + 2\sqrt{\frac{\gamma}{\gamma-1}\Delta Q}\left[1 - \left(\frac{p_x}{p_k}\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{6a}$$

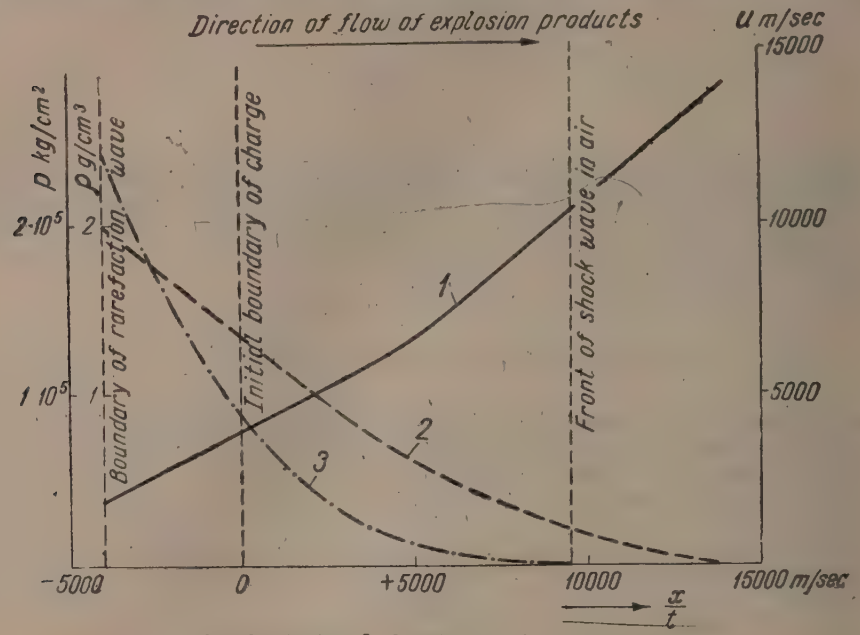
from which $p_x$, $u_x$, $T_x$ and $\rho_x$ may be determined.

The results of computations are given in the adjoined table.

| Explosive | $u_k$ | $u$ | $u_x$ | $p_x$ | $\rho_x$ | $T_x$ |
|---|---|---|---|---|---|---|
| Trotyl . . . . | 6000 | 11 300 | 7800 | 750 | 0.23 | 1200 |
| Picric acid . . | 6900 | 11 550 | 8000 | 800 | 0.21 | 1200 |
| Ten . . . . . | 7900 | 18 500 | 9800 | 1200 | 0.26 | 1600 |
| Tetryl . . . . | 7400 | 11 950 | 9200 | 900 | 0.25 | 1300 |

The temperature at the front of the shock-wave turns out to be of the order of 20 000 — 30 000° if the equation of state derived for an ideal gas is used,

and neither its dissociation nor ionization is taken into account. It is obvious that both these processes occur and considerably lower the temperature at the front of the shock-wave. Inasmuch as this is accompanied by considerable radiation, it is in general difficult to draw any conclusions regarding the nature of the shock-wave, and it may turn out that an abrupt shock-wave is not formed under these conditions. On the photographs made by Prof. G. I. Pokrovsky's method a shock-wave before the front of the gas is indeed absent.




*1*, velocity ($u$). *2*, density ($\rho$). *3*, pressure ($p$)

Let us carry the Riemann solution to the end, analysing separately both regions of the expanding detonation products and connecting the solutions at their point of conjugation. We have the following equations:

$$\frac{x}{t} = u - c, \tag{9}$$

$$u = u_{\text{initial}} + \frac{2c_{\text{initial}}}{n-1} - \frac{2c}{n-1} \text{ when } u_{\text{initial}} \leqslant u \leqslant \frac{3n-1}{n^2-1} D \tag{10}$$

$$u = \frac{3n-1}{n^2-1} D + \frac{2c_k}{\gamma-1} - \frac{2c}{\gamma-1} \text{ when } \frac{3n-1}{n^2-1} D \leqslant u \leqslant \frac{3n-1}{n^2-1} D + \frac{2c_k}{\gamma-1} \tag{10a}$$

Let us analyse any ideal explosive, closely resembling ten in its properties assuming that $n=3$, $\gamma=1.3$, $D=8000$ m/sec., $Q=1400$ cal./g and $\rho_0=1.5$. The results of the computation are given in the accompanying figure.

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Received
24. VII. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*PHYSICAL CHEMISTRY*

# INTERACTION OF HYDRATED FATTY ACID CRYSTALS WITH ELECTROLYTES, AND THEIR POLYMORPHISM AND EQUILIBRIUM WITH MONOLAYERS

By A. A. TRAPEZNIKOV

*(Communicated by A. N. Frumkin, Member of the Academy, 26.VIII.1944)*

Earlier (1) it was found by the author that when crystals of higher aliphatic compounds come in contact with water, the latter penetrates them and interacting with the organic substances, forms a hydrate. In the case of alcohols this even leads to a 2—3° rise in the melting points of the crystals, which is an unusual occurrence to the rule and indicates a marked change in their energy state. Fatty acid crystals are likewise «impregnated» by water to hydrates, although in this case the melting point is lowered somewhat.

While experimenting with fatty acid crystals a new phenomenon was discovered, namely, that electrolytes penetrate hydrated crystal and are adsorbed on the polar groups of the internal planes of the lattice.

The phenomenon of hydration and penetration of crystals by electrolytes is of specific interest in studying the reactions taking place in three-dimensional crystals of organic substances, and is also to be taken into account in studying the equilibrium between a three-dimensional phase and a monolayer. Heretofore, however, in the investigations of Cary and Rideal (2), and Frumkin and Pankratov (3), for example, who studied the two-dimensional pressure of a monolayer in equilibrium with crystals, no account at all was taken of the possible occurrence of the described effect and, hence, of the possible change in the energy state of the crystal.

To investigate the thermal properties of hydrated crystals and their interaction with electrolytes a new method was employed consisting in measuring the temperature dependence of the two-dimensional pressure of a monolayer in equilibrium with a crystal.

The influence of electrolytes makes itself felt both on the equilibrium state and on the kinetics of the formation of a monolayer.

The kinetics of the formation of a monolayer from a crystal. In Fig. 1 are given the curves of the change in the two-dimensional pressure $F$ as a function of the time $\tau$ that crystals of palmitic acid* remain on the surface of an aqueous solution of the electrolyte. Although the «perimeter of evaporation» of the solid phase does not remain the same in individual experiments in which identical quantities of the substance were sprinkled onto the surface, it is clear that the difference in the rates of formation of a layer depends in the main upon the nature of the underlying liquid. This undoubtedly refers to the limiting pressure, regardless of the number of crystals.

* The palmitic acid («Kahlbaum») was purified by recrystallizating it five times from ethyl alcohol; the melting point rose until it reached the constant value of 62.2°C.

The equilibrium pressure of a monolayer is lower on a 0.047 $N$ HCl solution than on water, and much higher on a 0.01 $N$ NaCl solution. The rate of formation of a monolayer on a NaCl solution is also noticeably higher than on $H_2O$.

For the indicated liquids in bulk the kinetics curves $F = f(\tau)$ have a normal course of continuously increasing pressure, reaching the limiting equilibrium value.

For a 0.02 $N$ $CaCl_2$ solution a new and anomalous course of the kinetic curve was discovered. In this case (Fig. 1, *а* and *б*) the pressure at first increases very

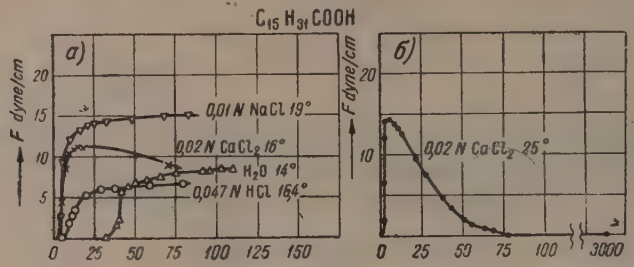


Fig. 1

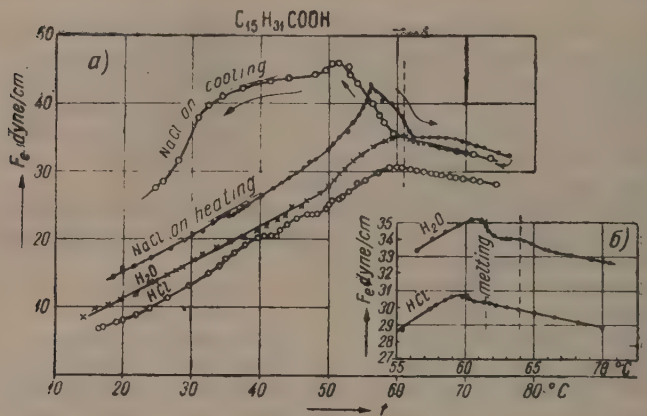


Fig. 2

rapidly, reaches a maximum and then spontaneously falls. The shape of the curve and the distinctness of the maximum depend, evidently, upon the temperature, the pH of the solution and the concentration of the electrolyte.

The temperature dependence of the two-dimensional pressure. After it has reached a constant pressure, the solution is heated at the rate of 0.1° a minute, and at the same time the change in the pressure $F_e$ is recorded.

The curves $F_e = f(t)$ for palmitic acid on $H_2O$, 0.047 $N$ HCl and 0.01 $N$ NaCl for the direct course—heating up—and the reverse course—cooling down (for NaCl)—are given in Fig. 2.

If the underlying liquid is acidulated, the equilibrium pressure is lower at all temperatures (including those lying above the melting point), as compared to neutral liquid. On a solution of NaCl the pressure is much higher than on water, primarily for the temperature region of the crystal state, *i. e.* below the melting point.

On the linear portions of the $F_e = f(t)$ curves there are a number of breaks. Thus, with the acid solutions in bulk breaks were observed at $t = 40 - 42.5°$ and $t = 47.5 - 50°$. On the curves for $H_2O$ and the NaCl solution the break is at $t = 50°$.

At this temperature the crystals of the acid settle deeper into the water, which is connected with an increase in their density, *i. e.* with a change in structure. Furthermore, on the $F_e = f(t)$ curve for 0.01 $N$ NaCl there is a practically unnoticeable pressure drop of 0.5 dynes at $t = 35°$. At this moment the crystals of the acid, arranged in groups over the surface of the solution, rapidly scatter over the entire surface singly.

The cause of this may be an abrupt change in state of the substance in bulk or, more likely, of the monolayer.

Of great interest is the cooling curve on the NaCl solution with three distinct portions below the solidification temperature of 60° and the unusually high equilibrium pressure.

Anisotropic-liquid hydrates. When crystals of palmitic acid were melted on aqueous solutions, not transparent lenses of the acid were observed to form as would have been the case if anhydrous crystals had been allowed to melt, but just as in the case of alcohols, turbid lenses of anisotropic-liquid hydrates were also observed. This likewise proves the penetration of water into the crystals.

Corresponding to the anisotropic-liquid state are some definite portions of the $F_e = f(t)$ curve (Fig. 2, magnified), although with acids they are less distinctly expressed and therefore more difficult to fix than with alcohols (1).

On HCl and $H_2O$ the lens remains turbid within the temperature angle of 61.5 to 66°.

The turbid lenses forming on HCl are irregular in shape, resembling the arrangement of the crystals on the surface. Evidently, the increased viscosity of the anisotropic-liquid hydrate I hampers the rounding off of the lens, which sets in only about 63.8° with partial decrease in turbidity, probably owing to transformation into the anisotropic-liquid hydrate II.

Discussion of results. First of all, all the observed phenomena indicate that the $F_e = f(t)$ curves refer to equilibrium with the hydrated crystals and not with the anhydrous, as Cary and Rideal assumed. The course of curves likewise differs from that found by these authors. Thus, the break in the curve (at maximum pressure) does not coincide with the visible melting point of the crystals. In connexion with this the latent heats of transformation, which may be computed from the slopes of the curves, will likewise differ, which is connected with the hydration of the crystals. The breaks in the curves may be the consequence of transformation not only in the substance in bulk, but also in the monolayer. This requires further investigation. However, the temperature of 50°C for visible submersion of the crystals in the water is very close to the temperature of 48 or 50° for the transformation of the β- or γ-modifications into the α-modification of anhydrous crystals of palmitic acid, that was found by Dupré la Tour (4). Therefore it may be assumed that the transformations in hydrated crystals of the acid have much in common with the polymorphous transformations in anhydrous crystals.

Evidently, the unusually high pressure of the cooling curve as compared with the heating curve on the NaCl solution is connected with the polymorphous hydrates, *i. e.* with the fact that the modifications formed upon cooling differ from that formed upon heating.

It is interesting that the state of the crystals determining the magnitude of the equilibrium pressure depends upon the acidity of the underlying liquid. The latter, penetrating into the crystal, affects the ionization of the polar group of the fatty acid molecule in the crystal and its degree of hydration.

The anomalous course of the kinetics curve $F = f(t)$ that was found for the $CaCl_2$ solution may be explained only by the interaction of the $Ca^{++}$ ions with the molecules of the acid in the crystal, but not in the monolayer. Evidently, the left part of the curve of the accelerated increase in pressure of the monolayer immediately after the crystals come into contact with the $CaCl_2$ solution is connected with the intensified penetration of the solution into the crystal, whereby the distances between the molecules of the crystal widen and the bonds weaken. The decrease

in pressure with time following this (possibly to zero, the curve in Fig. 1, *б*) is connected with the strong interaction of the $Ca^{++}$ ions with the fatty acid molecules that sets in, and with the formation of a calcium soap possessing a greater energy of interaction and, consequently, a higher energy of transition of the molecules into the monolayer.

In connexion with this, the molecules that broke off earlier, forming a monolayer, return to the crystal, and the pressure falls.

Evidently, at not high temperatures the $Ca^{++}$ ions penetrate the crystal only to a small depth. Adsorption of the $Ca^{++}$ ions on the layers of the molecules of the fatty acid adjoining the surface of the crystal hampers the further penetration of the crystals by the ions.

It is possible also that the compression of the surface layers of the lattice under the influence of interaction with $Ca^{++}$ ions initiates the transformation and leads to the formation of new, more stable polymorphous modifications of the entire crystal of the acid.

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26. VIII. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*PHYSICAL CHEMISTRY*

# ON THE TEMPERATURE OF SELF-INFLAMMATION OF A COAL SURFACE WASHED BY A GAS STREAM

By A. B. CHERNYSHEV, Corresponding Member of the Academy, and L. M. LIUBINSKAYA

Self-inflammation temperature is an important characteristic of coal, as is its reaction ability, the melting point of its ash, *etc.* This characteristic acquires a special significance in connexion with the underground gasification of coal.

In aboveground gas producers the coal moves towards an air blast issuing from a stationary aperture. In underground gasification, on the other hand, the coal seam is immobile and therefore, in order to gasify it uniformly, the point at which the blast is introduced has to be shifted periodically. This results in an abrupt shift of the combustion zone. Formation of a new combustion zone after the blast has been reversed takes place on a sector of the seam where the temperature conditions ensure the self-inflammation of the coal surface. A knowledge of these conditions is quite essential for controlling the process of underground gasification of coal. The visible inflammation of the coal arises from reactions of oxidation and pyrogenetic decomposition, and does not coincide with the start of the inflammation process.

Owing to the complexity of this process, different opinions as to the criterion determining the inflammation temperature have existed among various investigators. Perhaps the most acceptable is the following definition formulated by van't Hoff: «The inflammation temperature is attained when the initial heat loss due to thermal conductivity, *etc.*, becomes equal to the heat generated by the transformation in the same time interval». N. N. Semenov was the first to describe the boundary conditions of self-inflammation in mathematical terms as the equality between the loss of heat and the evolution of heat in the course of reaction (1). W. Gumz (2) modified van't Hoff's definition replacing the term «inflammation temperature» by that of «inflammation region», by which is to be understood the temperature region within which the reaction velocity noticeably increases and the heat generated not only compensates the heat loss, but is sufficient to produce further increase in the reaction rate, which leads to a rise in temperature and, finally, a visible inflammation of coal. It follows from the formulations of van't Hoff and Gumz that the inflammation temperature may change within definite limits, depending on the thermal conditions under which inflammation occurs.

The existing methods for determining the inflammation temperatures may be divided into three main groups: a) at a constant temperature of the surrounding medium (Moore, Lange, Masson and Hamilton, and others); b) at a changing (rising) temperature of the medium (R. Wheeler, Bunte, Nelson, and others); c) adiabatic (Davis and Byrne, Schäper, Makarov, and others). We adopted the second method.

According to the standard procedure employed by Wheeler ([3]), the point of intersection of the mean temperatures of a coal sample and the furnace walls is taken as the «minimum critical inflammation temperature». In our experiments the inflammation temperature was determined in the following manner: a piece of coal was placed into a specially built apparatus; over its surface a free channel was left along which a heated blast was introduced which ignited the coal.

In this case the minimum critical inflammation temperature (after Wheeler) could be evidently taken as the point of intersection of the temperature of the coal

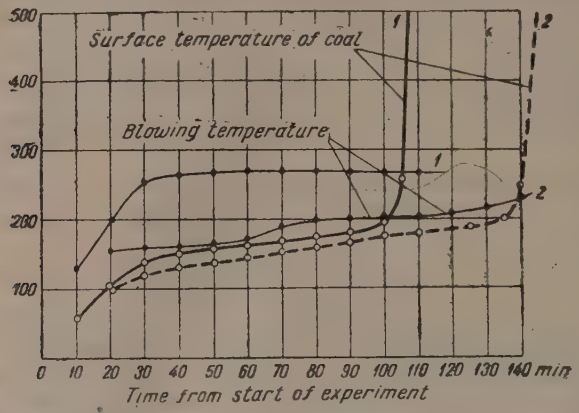


Fig. 1. The change in the temperature of the surface of Moscow coal washed by a stream of heated air

surface and the temperature of the heated blast. However, the experiment showed that the position of the point of intersection for the same coal varies greatly depending upon the experimental conditions; at the same time a definite narrow temperature region was established for every kind of coal, within which, in spite of considerable changes in the experimental conditions, sharp bend in the curve for the temperature of the coal surface was obtained in all experiments characterizing the spontaneous increase in the reaction rate, rapidly bringing on a visible inflammation of the coal (Fig. 1).

Various types of coals from the Donez and Moscow coal basins were investigated. Ignition regions in the air blast found experimentally are given in Table 1.

Table 1

| Coal variety | Inflammation temperature range (in °C) |
|---|---|
| Moscow | 180—200 |
| Donbas «D» | 260—280 |
| Donbas «G» | 290—310 |
| Donbas «PJ» | 350—370 |
| Donbas «K» | 410—430 |
| Donbas «PS» | 460—480 |
| Donbas anthracite | 500—520 |

There is practically a linear dependence of the mean inflammation temperature upon the content of volatile matter in the coal.

The change in the conditions of heat transfer (the several-field change in the velocity of the air stream washing the coal surface, the change in the temperature and the rate of heating of the air, and so forth) noticeably affects only the rate of inflammation of the coal (Fig. 2).

With an increase in the velocity of the heated air blast the rate of inflammation of the coal increases, owing to a more effective heat transfer. This factor becomes conspicuous with a flow of distinctly turbulent nature (experiment No. 3).

A considerable change in the range of inflammation temperature occurs if: a) the coal has been subjected to preliminary heat treatment; b) the composition of the blast is changed (the oxygen concentration is changed).

In the case of preliminary heat treatment of the coal, the inflammation temperature of the coke residue rises. Thus, for example, the ignition region of the semi-coke residue obtained from Moscow coal ($V_{\text{п}} = 8$—10 per cent) was found to be 330—350° C instead of 180—200°C for the original specimen of coal.

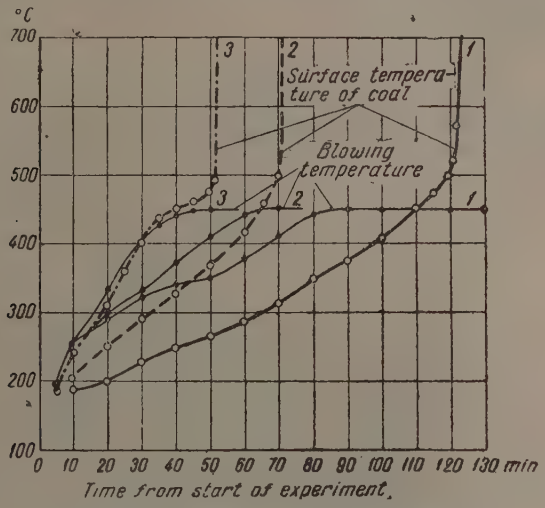


Fig. 2. The change in the rate of inflammation of coke from coal «K», depending upon the rate of flow of the heated air. *1*, the rate of flow 1 m/sec.; *2*, the rate of flow 3 m/sec.; *3*, the rate of flow 6 m/sec.

An increase in the oxygen content of the blast lowers the inflammation temperature of the coal. In Table 2 experimentally determined inflammation regions for coke obtained from Donbas coal «K» are listed against different oxygen concentrations in the blast.

Table 2

| Oxygen concentration in the blast (in %) | Inflammation temperature range (in °C) |
|---|---|
| 15 | 530—540 |
| 21 | 480—490 |
| 35 | 430—440 |
| 50 | 380—390 |
| 60 | 350—360 |

Thus, upon increasing the oxygen concentration in the blast to 35 per cent one may in the given case make the coke inflame exactly at the same temperature as the original «K» coal specimen. This is quite essential, since when the blast is reversed in a coal seam undergoing underground gasification, we have to do with coal that has already been partly pretreated thermally.

The increase in the oxygen content in the blast not only lowers the temperature inflammation, but also increases the rate of inflammation of the coal.

Conclusions. As a result of the experiments carried out it has been established that when gas of a given composition washes a coal surface, every type of coal is characterized by a definite «range of inflammation» temperature within which the reaction is strongly accelerated, and ends in visible inflammation of the coal.

By changing the amount and temperature of the blast one may achieve a noticeable change in the rate of inflammation of the coal surface in the same temperature range. Upon changing the composition (oxygen concentration) of the blast one may change not only the rate of inflammation but also the temperature, *i. e.* the spot where a new combustion zone is formed in the coal seam undergoing underground gasification after the blast has been reversed.

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Received
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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*BIOCHEMISTRY*

# MUSHROOMS AS A SOURCE OF VITAMIN PP

By N. I. PROSKURIAKOV and O. A. PAVLINOVA

*(Communicated by A. N. Bach, Member of the Academy, 20.IX.1944)*

The important rôle played by nicotinic acid in the nutrition of man and particularly in the treatment of pellagra served as a stimulus to numerous studies dealing with the determination of the content of this vitamin in the most diverse substances of both animal and vegetable origin. Along with this, methods for the determination of nicotinic acid were devised and improved.

In USA and in several other countries nicotinic acid is being produced synthetically in considerable quantities. Its addition to food, for example to white wheat flour, has been practised in baking industry since 1941 and is encouraged as «a measure of National defense» (1).

At the same time in USA intensive and systematic research is carried on along the line of quantitative determination of nicotinic acid in various natural products entering in the common diet. According to data by Cheldelin and Williams (2), the average daily dose of nicotinic acid is 11 mg.

In order to compare the percentage content of nicotinic acid in different foodstuffs some examples will be cited here. Thus, out of the large number of objects analysed by them for the content of nicotinic acid, Teply, Strong and Elvehjem (3) found vegetables and fruit to be poor in nicotinic acid. Tomatoes (16.5 mg per cent to dry matter) and carrots (14.0 mg per cent) have been found to be the highest in this vitamin; while other vegetables have been found to contain considerably lesser amounts of nicotinic acid. Fruits proved to be still poorer in it: apples, for example, contain 3.54 mg per cent, grapes 1.8 mg per cent, *etc.* As to cereals, these are considered rather rich in the vitamin discussed; wheat and rye contain about 4—5 mg per cent of it. In milk and eggs nicotinic acid is present in most insignificant amounts only. Yeast has been found to be the richest source of nicotinic acid, its content varying from 16 to 61 mg per cent (to dry matter) according to the origin of the product (data by Bandier and Hald (4)). With the aid of special treatments and concentration extracts with a content of nicotinic acid from 180 to 220 mg per cent are produced from the original yeast.

Determinations of nicotinic acid were made with the aid of various chemical and microbiological methods. The latter are based on the use of specific microorganisms highly responsive to the presence of definite amounts of nicotinic acid in the medium. Both groups of methods, although differing in principle from each other, yield closely similar results, as shown by Greene, Black and Howland (5). Microbiological methods should be supposed to approach more closely the biological tests than chemical methods do.

In this country the production of pure nicotinic acid from nicotine as a waste product of the makhorka industry has also been developed on a large scale.

It seems to us very important, however, to look for new natural raw material rich in nicotinic acid. And it seemed not unlikely that widely distributed raw material will be found that would need no complex preliminary treatment to make it edible.

The authors succeeded in finding out that such wild-growing plants satisfying to the above requirements are common edible mushrooms, growing in large quantities in many regions of the USSR. In foreign literature no data are available concerning the determination of nicotinic acid in mushrooms. Only Willstaedt (6) mentions that some mushrooms contain considerable amounts of thiamine and riboflavin: *e. g. Lactarius deliciosus* contains 37 international units of thiamine and 690 γ of riboflavin; the field champignon (*Psalliota campestris*), 640 γ of riboflavin, *etc.* Thus, there is information to the effect that edible mushrooms are attractive and promising objects to be tested for their content of vitamins of the B complex.

In the present research we made use of the method of Bandier and Hald, developed by these authors for a determination of nicotinic acid in yeast and other products of organic origin.

Like most other chemical methods, this method is based on the well-known reaction of Koenig, which consists in the following: pyridine, upon an interaction with bromo-cyanogen and an aromatic amine (primary or secondary) produces a definite colour according to the amine used. The material to be tested is in the first place subjected to extraction by means of acid or alkaline hydrolysis, with a subsequent purification and treatment with bromo-cyanogen at high temperature, followed by an addition of *p*-methylaminophenol-sulphate (methole). The resulting yellow colour of the solution is evaluated after one hour's stay in the dark with the aid of Pulfrich's photometer (S 43 filter). The control is tested with the same reagents but without the addition of bromo-cyanogen. The content of nicotinic acid is evaluated by the difference in the indices of extinction between the basic and the control experiments.

Table 1 presents data on the content of the nicotinic acid in different dried yeasts*.

Table 1

Nicotinic Acid Content in Various Dried Yeasts (in mg per cent to air-dry matter)

| | |
|---|---|
| American dried yeast «Maca» | 10.40 |
| Bakery yeast from the Moscow yeast plant | 18.75 |
| Brewer's yeast, strain 776 | 21.87 |
| *Monilia murmanica*, grown on molasses | 27.23 |
| American bakery active yeast of Fleischman | 29.70 |
| *Monilia murmanica* grown on a woody hydrolysate | 37.50 |

The specimens of yeasts analysed were found to contain about the same amounts of nicotinic acid as those reported by other authors; thus, in addition to the data cited above, Green, Black and Howland found in various yeasts from 20 to 65 mg per cent, and Kringstag and Thoresen (7) from 25 to 45 mg per cent of nicotinic acid. The amounts of nicotinic acid discovered by the present authors in the mushrooms are given in Table 2. The material analysed comprised both pileus and stipe of the mushroom.

*Oidium lactis* was used as a representative of imperfect mushrooms; it was grown upon milk serum. Other edible mushrooms were collected in the Moscow region and dried in the usual manner. Previous to determination they were subjected to additional drying, and then carefully ground in a mortar. *Boletus edulis* were collected in two localities far removed from each other.

* The authors' thanks are due to Prof. R. A. Plevako for placing at their disposal the yeast specimens to be tested.

Table 2

Contents of Nicotinic Acid in Various Mushrooms (in mg per cent to dry matter)

| | |
|---|---|
| *Oidium lactis* | 13.74 |
| *Armillaria mellea* | 34.15 |
| *Boletus versipellis* | 36.14 |
| *Boletus bovinus* | 49.75 |
| *Cantharellus cibarius* | 50.21 |
| *Boletus scaber* | 63.13 |
| *Boletus edulis* I | 71.64 |
| » » II | 75.44 |

From the data obtained it may be inferred that edible mushrooms are as it were natural concentrates of vitamin PP, their content in this vitamin exceeding even yeasts which have until now been considered as objects very high in this vitamin.

It is therefore necessary to undertake extensive investigation of the fungal flora, including the mushrooms of the north of the USSR, where the need in natural vitamin concentrates is especially great. At the same time more rational methods for the treatment of mushrooms should be devised, permitting of a maximum extraction of nicotinic acid and of making it more readily soluble and assimilated by the organism.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*GEOLOGY*

# WATER CONTENT IN BOTTOM DEPOSITS OF THE CASPIAN SEA

By S. W. BRUJEWICZ

*(Communicated by W. I. Vernadsky, Member of the Academy, 21.VII.1944)*

Sampling of natural marine sediments as practised in the course of geological investigation of the sea-bottom is usually accompanied by a mere description of the samples taken, the latter not being subjected to laboratory analyses until they are dried. As a matter for regret, this holds true even of such subtle determinations as are those of the mechanical composition of sediments. As a result, oceanographic literature contains scarcely any data bearing on such an important question as that of the determination of water content in marine sediments under natural conditions. Corresponding records taken by the Atlantic Expedition on board ship «Meteor» in 1925—1927 ([2–3]) were subjected to no special discussion either*, being no more than subsidiary determinations with that. The characteristic of water content in the sediments, both in its upper layers and in the beds by which these are underlain, is of prime importance, however. It is of great use when determining the mineral and organic substances accumulated by the sediments, as well as when estimating the possibility of these sediments having been washed-off, their relative age, and characterizing the mechanical properties of the bottom deposits, in the first line, its viscosity, which is of great practical importance.

The material described below, in the collection and analytical treatments of which A. I. Belev and E. G. Vinogradova took part, refers to samples taken with the Petersen bottom sampler (the upper layer up to 20 cm thick) and with the Ekman tube yielding monoliths up to 1 m long. Directly upon sampling the material was placed in glass jars, stoppered with corks and carefully embedded in paraffin or resin. Samples weighing about 15 g dried at 105—110°C.

Summing up the data of Table 1 for different regions concerning 71 samples, the following values** for the upper layer of various types of sediments for the sea as a whole are obtained: mud, 65 per cent (16 samples); sandy mud, 38 per cent (8); muddy sand, 23 per cent (5); sand, 19 per cent (3); mud with shells, 50 per cent (6); shells, 30 per cent (12); muddy sand with shells, 25 per cent (12); sand with shells, 21 per cent 4).

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* Besides, the ground samples being to small (0.5 g), the results of these determinations should be considered not without qualification.

** The mean values have been calculated in proportion to the number of samples taken for each region of the sea.

Table 1

Average Values of Water Content of the Upper Layer of Bottom Deposits of the Caspian Sea in per cent to Weight of Natural Ground for Different Regions of the Sea (sampling done with a bottom sampler; figures in parentheses indicate number of samples)

| Place of sampling | Nature of bottom deposits and average water content | | | |
|---|---|---|---|---|
| Northern part of the Caspian Sea, May, 1939 | mud<br>73.4 (1) | sandy mud<br>29.4 (6) | muddy sand<br>23.9 (3) | sand with shells<br>21.4 (4) |
| Do, May, 1940 | mud<br>60.9 (5) | sandy mud and silty sand<br>33.7 (5) | muddy sand with shells<br>24.6 (12) | shells<br>27.5 (5) |
| Tyub-Karagan Bay, autumn, 1939 | mud<br>72.2 (3) | sandy mud<br>47.8 (2) | muddy sand<br>22.5 (2) | sand<br>19.2 (3) |
| Middle part of the Caspian Sea, summer, 1940 | mud<br>64.7 (2) | mud with shells<br>49.9 (4) | — | shells<br>28.9 (3) |
| Southern part of the Caspian Sea, summer, 1040 | mud<br>64.8 (2) | mud with shells<br>49.3 (2) | — | shells<br>30.9 (4) |
| Krasnovodsk Bay, autumn, 1935 | mud<br>66.1 (3) | — | — | — |

For sediments losing water while sampling is done (sand, shells), the value of water content in the sediments was put at its total moisture capacity equal to that of sediments *in situ*. The distribution of sediments according to the mechanical composition was made on the basis of the following characteristics (1): sand, less than 5 per cent of the fine fraction (particles smaller than 0.01 mm); muddy sand, 5 to 10 per cent; sandy mud, 10 to 30 per cent; mud, 30 to 50 per cent; clayey mud, over 50 per cent of the fine fraction. It will be seen that the moisture content in sediments is ultimately related to the mechanical composition; at the same time

Table 2

Water Content in the Upper Layer of Muds at Different Depths of the Caspian Sea

| Region of sea and station | Depth of sea, m | Water content, % | Tool used for sampling |
|---|---|---|---|
| Northern part of the Caspian Sea, 1939—1940, mean . . . . . | about 6 | 67.2 | Bottom sampler |
| Middle part of the Caspian Sea, 1940 . . | 19—35<br>mean 27 | 64.7 | » |
| Southern part of the Caspian Sea, 1940<br>Section along Kurinsky Kamen—Ogurchinsky island | | | |
| Station Гх 29 (eastern part) . . . . . | 85 | 66.4 | »<br>0 to 5 cm |
| » Гх 28 » » . . . . . | 460 | 64.9 | Ekman tube<br>0—10 cm |
| » Гх 26 » » . . . . . | 960 | 69.5 | Ekman tube |
| Mean for the whole Caspian Sea . . . . . | — | 65.5 | — |

the data presented do not point to the existence of any relation between the moisture content in the upper (0 to 10, 0 to 20 cm) layer of bottom deposits and the depth of the sea at a given place. This is easily observable in muds which occur at any depth of the sea. As may be seen from Table 2, the differences in water contents in mud at depths ranging from a few metres and the maximum depth (960 m)do not exceed the difference in the water content values for neighbouring stations, taken at a nearly identical depth.

The only exception are the muds of the section along Zhiloy island—Kuuli cape (Table 3), passing along the submarine Apsheron Ridge. When studying the distribution of moisture in natural sediments along the vertical, we have to content ourselves with data on muds for lack of suitable implements for collecting undisturbed monoliths of solid sediments (sand and shells). The distribution of water content in muds along the vertical is of much greater interest than the differences in water content of the upper layer for different regions of the sea. As may be seen from Table 3, the water content in the mud sediments in different regions of the sea varies considerably, and the character of the downward decrease in water content is different as to the region, too. The most interesting data are obtained when comparing water content in the most deep-water sediments of the southern part of the Caspian sea (station 26, depth 960 m, station Гх 43, station 28 with depths of 450—460 m) with shallow-water mud deposits.

As seen from Table 3, determinations of water content in sediments sampled at deep-water and shallow-water stations yield values closely approaching one another, irrespective of the sharp differences in the depth of the sea, ranging between a few metres and nearly one kilometre. To these depths correspond widely differing rates of sedimentation, which fail, however, to affect to any considerable extent the values of the water content of the lower strata of the sediments. Thus, the age of such sediments as had been subject neither to redeposition nor to washing, had no essential influence upon the value and distribution of the water content along the vertical. The decrease in the water content of normally deposited muds is usually connected with the presence of inclusions—shells, pebbles.

Exceptional interest belongs to the unusually low water content in the clayey muds in the section Zhiloy island—Kuuli Cape, passing along the submarine Apsheron threshold, dividing the depressions of the middle and southern parts of the Caspian Sea, and especially at station 18, lying deep within the gutter cutting this threshold. The unusually low water content observed in this very viscous fine clay (31.1—29.1 per cent) beginning from the topmost horizon, points to the fact that here we have to do with older sediments exposed as a result of erosion rather than with normally deposited sediments (along the vertical). The intense currents in the Apsheron gutter serving as a channel-way for water exchange between the depressions of the middle and southern parts of the Caspian Sea seem well to substantiate the statement made above. From the occurrence in this section of a patch of pebblestone, surrounded by sandy muds referred presumably to the time of the productive formation, V. P. Baturin and E. K. Kopylova also speak of an erosion of the sediments and exposure of ancient deposits.

The basic conclusions which may be inferred from the material presented above are as follows:

1. The moisture content in natural marine sediments is intimately related to the mechanical composition of the sediments: the highest moisture content corresponds to the finest mechanical composition (muds, clays); the lowest moisture content, to sands.

2. Depth and local features of the water body within the given type and mechanical composition of sediments (mud) have no influence upon moisture content in the upper (0—10, 0—20 cm) layer of ground, which is almost equal everywhere and at any depth.

3. The vertical distribution of moisture content within the bottom deposits is determined by the type of the sediment (mechanical composition, inclusions) rather than by the depth of the sea. This is true only of the normally deposited, non-eroded sediments.

## Distribution of Water Content in Muds of the Caspian Sea along the Vertical (water content in per cent

| Tyub-Karagan Bay, October—November 1939 | | | | | Northern part of the Caspian Sea, Ju | | | |
|---|---|---|---|---|---|---|---|---|
| Sta. Гх 84, depth 6.5 m, angle of bay, I. X. 1939 | Sta. Гх 92, depth 6 m, angle of bay, 24. VI. 1940 | Sta. Гх 83, depth 7 m | Sta. Гх 85, depth 8.5 m | Sta. Гх 90, depth 10.5 m | Sta. 19, depth 5.1 m | Sta. 44, depth 4.3 m | Sta. 49, depth 4.0 m | Sta. 47, d 8.3 m |
| 0—17 cm 77.0% | 0—5 cm 75.6% | 0—7 cm — | 0—5 cm — | 0—15 cm 46.0% | 0—5 cm 69.1% | 0—5 cm 73.4% | 0—5 cm — | 0—5 cm |
| 17—20 — | 5—10 61.1 | 7—15 69.3% | 5—10 — | 15—25 41.3 | 5.10 46.1 | 5—10 69.9 | 5—10 45.0% | 5—10 |
| 20—40 60.0 | 10—20 62.6 | 15—30 57.3 | 10—20 46.3% | 25—40 42.9 | 10.27 26.6 | 10—20 67.8 | 10—20 35.9 | 10—20 |
| 40—50 45.2 | 20—30 51.5 | 30—40 40.5 | 20—30 41.1 | 40—50 38.8 | | 20—30 44.1 | 20—30 35.3 | 20—30 |
| 50—83 32.8 | 30—40 45.0 | 40—50 40.1 | 30—40 38.4 | | | | 30—40 41.0 | 30—40 |
| | 40—50 37.6 | 50—60 38.6 | 40—50 40.3 | | | | 40—50 41.0 | 40—50 |
| | 50—60 37.7 | 60—70 38.5 | 50—67 35.8 | | | | 50—56 51.8 | 50—64 |
| | 60—70 37.5 | 70—80 40.9 | | | | | | |
| | 70—80 39.2 | 80—90 41.1 | | | | | | |
| | 80—90 40.0 | | | | | | | |
| | 90—100 38.2 | | | | | | | |
| | 100—106 36.0 | | | | | | | |

| Middle part of the Caspian Sea, April—May, 1940 | | Section Zhiloy island—cape Kuuli, May, 1940 (submarine Apsheron Ridge) | | | Bed of Southern part of the Caspian Sea, section Kurinsky Kamen—Ogurchinsky island, 1940 | Eastern slope of the Southern part of the Caspian Sea | | |
|---|---|---|---|---|---|---|---|---|
| Sta. 1, depth 19 m | Sta. 130, depth 258 m | Sta. 16, depth 100 m | Sta. 17, depth 171 m | Sta. 18, depth 200 m | Sta. 26, depth 960 m | Sta. Гх 43, depth 450 m, December, 1936 | Sta. 28, depth 460 m, June, 1940 | Sta. K2, d 5.2 m |
| 0—5 cm 81.5% | 0—5 cm 69.3% | 0—5 cm 48.2% | 0—5 cm 42.8% | 0—5 cm 31.1% | 0—10 cm 69.5% | 0—20 cm 69.2% | 0—5 cm 64.9% | 0—20 cm |
| 9—10 70.5 | 5—15 66.1 | 5—10 43.9 | 5—16 42.2 | 5—15 30.6 | 10—20 61.8 | 20—40 60.1 | 5—10 — | 20—30 |
| 10—20 63.9 | 15—25 61.1 | 10—20 35.2 | 15—33 37.0 | 15—30 29.1 | 20—30 61.5 | 40—60 53.6 | 10—20 60.4 | 30—55 |
| 20—30 64.6 | 25—35 61.8 | 20—38 30.5 | | | 30—40 61.1 | 60—80 51.0 | 20—30 — | |
| 30—40 60.4 | 35—45 50.4 | | | | 40—50 55.5 | 80—100 46.9 | 30—40 53.5 | |
| 40—50 63.3 | 45—60 52.6 | | | | 50—60 58.2 | — 50.5 | 40—50 52.1 | |
| 50—64 56.9 | | | | | | | 50—60 51.0 | |
| | | | | | | | 60—70 45.5 | |

Table 3

natural grounds)

July, 1940

| h | Sta. 48, depth 7 m | | Sta. 57, depth 3 m | |
|---|---|---|---|---|
| 9% | 0—5 cm | 79.8% | 0—5 cm | 65.3% |
| 3 | 5—10 | 75.1 | 5—10 | 66.6 |
| 6 | 10—20 | 69.9 | 10—20 | 56.8 |
| 2 | 20—30 | 64.3 | 20—30 | 53.3 |
| 3 | 30—40 | 57.5 | 30—40 | 56.2 |
| 9 | 40—50 | 52.0 | 40—50 | 43.9 |
| 9 | 50—58 | 48.6 | 50—60 | 37.5 |
| | | | 60—75 | 37.4 |

snovodsk bay, November, 1935

| th | Sta. K4, depth 5.5 m | | Sta. K7, depth 9.5 m | |
|---|---|---|---|---|
| 7% | 0—26 cm | 61.6% | 0—10 cm | 70.2% |
| 9 | 20—45 | 53.6 | 10—40 | 65.6 |
| 1 | 45—79 | 56.6 | | |

4. The deviation of moisture content from the normal for a given type of sediments towards a decrease bears evidence of erosion of the sediments and exposure of older deposits on the surface.

5. The exposures in the region of the Apsheron submarine gutter of ancient clays with a moisture content sharply below the normal one should be taken to mean that under the present morphological and hydrological conditions no sediments can be deposited at this place, and thus are evidence of a change of the bottom relief resulting from tectonic processes.

Received
21. VII. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*MINERALOGY*

# A CLASSIFICATION OF THE LUSTRES OF MINERALS

By A. G. BETEKHTIN

*(Communicated by A. E. Fersman, Member of the Academy, 29.V.1944)*

The great diversity in the physical properties of minerals is widely used for their identification. Since immemorial times these properties used to be established empirically. Owing to recent advances in physics and chemistry, crystal chemistry in particular, crystallography and mineralogy, the first serious attempts to attack those problems have been made. In the present paper the author discusses one of the most important diagnostic features of minerals, namely their lustre.

In mineralogy definite terms have long since been adopted to describe the lustres of definite minerals, such as vitreous, greasy, adamantine, nacreous, metallic, *etc.* Those terms fail, however, to be duly classified in any text-book of mineralogy. But if we have recourse to physics, the question is settled rather simply.

As well known, the light flow striking upon the mineral is partly reflected, the frequency of vibrations being unaffected. It is the reflected light that is responsible for the effect of lustre characterizing a mineral. The intensity of the lustre, *i. e.* the amount of reflected light, is the greater, the sharper the difference between the velocities of light when passing to a crystalline medium, *i. e.* the larger the refractive index of the mineral. The lustre does not depend on the colour of the compounds.

The refractive indices of the minerals known, it is no difficult matter to calculate the coefficient of light reflection for most of them with the aid of Fresnel's formula:

$$R=\left(\frac{n-1}{n+1}\right)^2$$

where $R$ is the index of reflection, and $n$ is the average index of refraction of the mineral with respect to air. Substituting in this formula a series of definite values of $n$, it is easy to represent graphically the dependence of the index of reflection (lustre) on the refractive index (Fig. 1). It will be seen that the curve has a minimum at $n=1$, which approaches closely the refractive index of air. Since the great majority of minerals have a refractive index above unity, the values of the reflection coefficient $R$, with which we are concerned, will lie on the right-hand side of this minimum.

The gradations of lustre intensities of minerals that have long since been established in a practical way, fit almost precisely into the following step scale:

1) V i t r e o u s lustres, distinctive of minerals with $n = 1.3$ to 1.9. Among these are: ice ($n = 1.309$), cryolite ($n = 1.34$ to 1.36), fluorite ($n = 1.43$), quartz ($n = 1.544$); then follow numerous halogen compounds, carbonates, sulphates, silicates and other oxygen salts; the series ends in such minerals as spinel ($n = 1.73$), corundum ($n = 1.77$) and garnets ($n = 1.84$).

2) A d a m a n t i n e lustres, characteristic of minerals with $n = 1.9$ to 2.6. As examples the following may be cited: zircon ($n = 1.92$ to 1.96), cassiterite ($n = 1.99$ to 2.09), native sulphur with an adamantine lustre on the planes of faces ($n = 2.04$), sphalerite ($n = 2.3$ to 2.4), diamond ($n = 2.40$ to 2.46), greenockite

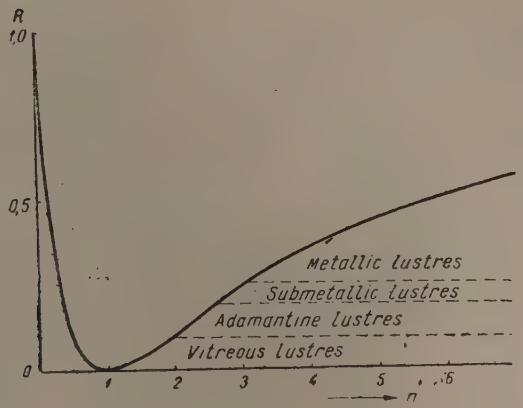


Fig. 1

($n = 2.5$), rutile ($n = 2.62$), the latter frequently possessing a submetallic lustre peculiar to the semi-transparent varieties.

3) S u b m e t a l l i c lustres of transparent and semi-transparent minerals with refractive indices (for a Li-light) $n = 2.6$ to 3.0. Examples: alabandite ($n = 2.70$), cuprite ($n = 2.85$), cinnabar ($n = 2.91$), hematite ($n = 3.01$).

4) M e t a l l i c lustres of minerals with refractive indices $n > 3$. In the order of increasing reflecting power the following examples may be cited: pyrolusite (crystalline), molybdenite, pyrrhotine, antimonite, galena, chalcopyrite, pyrite, aresenopyrite, bismuth, *etc.*

To the left of the minimum (Fig. 1) the curve of the reflecting power rises steeply. To this region with refractive indices less than unity fall only pure metals: silver ($n = 0.18$), gold ($n = 0.36$), copper ($n = 0.64$). It should be noted that in determining the reflecting power of opaque minerals, in addition to their refractive indices the coefficient of absorption, $k$, should also be considered. For these cases the index of reflection (for optically isotropic sections of minerals) is expressed by the following formula:

$$R = \frac{(n-1)^2 + n^2k^2}{(n+1)^2 + n^2k^2}$$

This means that for minerals with a metallic lustre (*i. e.* for the left and extreme right part of the branch of the curve of Fig. 1) the values of reflection coefficient will actually be somewhat higher than that obtained by Fresnel's formula. This fact, however, does not introduce any essential changes into the above scale of lustres. On the contrary, it goes well to account for the exceptions that seem to be rare. For example, magnetite has an index of reflection $2.42_{Na}$, *i. e.* it

must have an adamantine lustre; however, owing to its opacity, *i. e.* a considerable absorption of light, the reflection coefficient will be somewhat higher, passing in the diagram (Fig. 1) to the region of submetallic lustres. The reverse is true for auripigment which according to its refractive index ($n=2.81_{Li}$) should possess a submetallic lustre. Curiously enough, its finely crystalline aggregates actually have such a lustre; however, in fragments of large transparent crystals in fractures not along the cleavage a false, strongly adamantine lustre is observed.

If we turn to the question, what lustres predominate in the mineral kingdom, then, having arranged all the transparent and translucent minerals according to their mean refractive index (Fig. 2), we shall see a very distinctly pronounced wide maximum, for the values of $n=1.5$ to 1.7. The calculation shows that within group of minerals with a vitreous lustre about 70 per cent of the natural compounds with $n \leqslant 1.9$ are to be classed. Another group, embracing a smaller number of minerals, comprises those with a metallic lustre ($n>3.0$).

Another important factor influencing the result of light reflection is the nature of the surface from which the light rebounds. Above were discussed lustres of mineral, caused by smooth glassy surfaces (*i. e.* crystal faces and cleavage planes). But if at the fracture the mineral has fine ragged or pitted surface rather than an ideally smooth one, the vitreous or adamantine lustres acquire a somewhat dull tinge. In this case the reflected light is partly dispersed. Then a g r e a s y lustre is produced. This phenomenon will clearly be perceived if the change in the lustre at a fresh fracture of rock salt in humid air is observed. In some days the brilliant surfaces will appear as if covered with a very fine film of grease. This will be

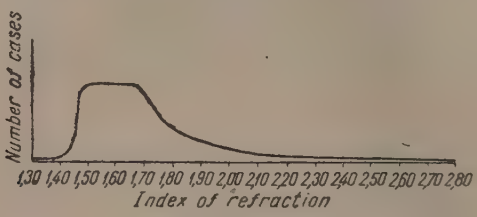


Fig. 2

especially apparent when compared to the surfaces of fresh fractures or completely weathered surfaces. The most typical examples of greasy lustre are presented by the lustre of native sulphur or of eleolite which has suffered a hardly perceivable decomposition.

The surfaces with a more strongly pronounced roughness possess a w a x y lustre. It is especially characteristic of crypto-crystalline masses and hard light-coloured gels. Such are, for instance, the frequently observed lustres of flints, of colloidal masses of minerals of the halloysite-garnierite series, *etc.* The same lustre characterizes crypto-crystalline, nearly dull aggregates of sphalerite, the finely aggregate structure of which can be proved only in polished sections under the microscope, by etching them with acids. In all of these cases the true lustre of minerals (vitreous or adamantine) is completely disguised.

Lastly, if the finely disperse masses have in addition a very fine porosity, their surface hardly reflects any light at all; or, better to say, light is dispersed in the most different directions. The microscopic pores are a kind of «pitfalls» for light The surfaces of this kind are named d u l l. As examples may serve chalk, kaolin (in the dry state), various ochres, sooty pyrolusite, *etc.*

For some minerals possessing a distinctly pronounced orientation of the elements of structure in one or two dimensions in space, peculiar phenomena connected with lustre are observed. In minerals with a finely fibrous structure (asbestos, nemalite, selenite, *etc.*) a characteristic s i l k y lustre is always observed. Transparent minerals of a lamellated structure (*i. e.* a distinctly pronounced perfect cleavage) very frequently have the so-called n a c r e o u s lustre (examples: muscovite, lamellar gypsum, talc, *etc.*). That the appearance of a nacreous lustre

is connected with lamination, is easily proved by placing one upon the other a number of cover or window glasses and looking at the whole pack from above. A peculiar lustre will actually be observed, very much resembling that of pearls.

Thus, the lustres of minerals should be classed into different groups, each of of them being due to definite physical causes. In the light of what has been said above, the idea that this criterion is a matter of personal judgment should be completely discarded.

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Received
25. V. 1944.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*MINERALOGY*

# X-RAY STUDIES OF PROCESSES OF GROUND FREEZING

By I. SEDLETZKY, M. SUMGHIN and A. MALOVICHKO

*(Communicated by V. A. Obruchev, Member of the Academy, 30.V.1944)*

1. The methods that have until recently been applied to the study of freezing and frozen grounds required that the specimen be destroyed at the stage to be investigated. Thus, in order to study a certain process, a series of specimens were to be destroyed successively at different stages. But since the preparation of perfectly identical specimens is not feasible, the course followed by any process in such specimens cannot be identical. The results of investigations would therefore yield an averaged idea of the process, and it is quite probable that some particular details of the process might escape observation.

The present authors made it their aim to develop a method which would permit of a continuous study of the dynamics of the process of freezing and thawing of grounds without necessitating the destroying of the objects to be investigated. Such a method, as it will be shown below, is presented by X-ray analysis with the aid of X-ray photographs and phase determination.

It should be noted that until now the X-ray method was not applied to the study of frozen grounds either in the USSR or abroad. The present paper is the first attempt in this field.

2. As one of the principal questions without answering which it is impossible to inquire into many phenomena of freezing and thawing, is that of the mode of migration of moisture, the present paper deals with this fundamental problem.

To investigation were subjected complex models of ground: Beskudnikov loam, Jurassic clay and sand. As the models were used the grounds which served to the experiments of Bazhenova who also prepared the specimens* and saw to their freezing. The ground mass was prepared according to a previously chosen moisture content from an air-dry ground, after which it was placed into special pasteboard moulds $115 \times 80 \times 1 \times$ mm thick. The moulds were sealed with hot paraffin; the moisture content was determined before and after the experiment. The specimens were kept in an desiccator with water.

After the first X-ray photograph, the thawed specimens were transferred to a freezing chamber and kept there at a temperature of $-10°$C during 24 hours. Then X-ray photographs of them were taken again in a frozen condition. After being photographed, the specimens were kept for 24 hours at room temperature for thawing, after which X-ray photographs were taken. Then the specimens were weighed to determine the decrease in moisture.

The X-ray photographs were taken on an iron radiation, the voltage being 35 kV, and the amperage 8 mA, time of exposure 5 min., and distance of the specimen from the tube, 1500 mm.

* The authors wish to express their sincere gratitude to A. Bazhenova.

3. The X-ray photographs of specimens of three rocks of a different mechanical composition (sand, loam, clay) in their original state give an idea of the macrostructure of the ground, the quality of the packing and the presence of inclusions. In the frozen condition these rocks showed the following picture in the X-ray photographs. The sand did not show any changes as compared to the initial X-ray photograph; the Beskudnikov loam showed numerous dark bands, especially closely spaced in the centre, representing fractures in the skeleton of the ground filled with ice, and very few small undulating interlayers. The Jurassic clay showed in the X-ray photograph, in addition to large ice intercalations located in the centre of the specimen and arranged in all directions, small undulating ice intercalations closely spaced near the surface of the specimen.

After thawing, the sand did not show any changes; the Beskudnikov loam retained the general arrangement of the dark bands, characteristic of the frozen condition. The Jurassic clay showed the disappearance of the small bands and their partial conversion into dark larger bands. Thus, the results of the experiments of the first run show that the X-ray method (X-ray photographs) gives some idea of the changes taking place in the internal structure of the specimens of grounds without destroying them. As a result of taking X-ray photographs of specimens at the three above-mentioned stages, it can be quite precisely established after which type (according to Teber's classification) the migration of moisture takes place when the specimen is congeled. The data go to confirm the experiments by Bazhenova [1].

The following observations are new: the different nature of the fracturing in the Beskudnikov loam and the Jurassic clay make us assume that this method will reveal still other details in the process of migration dependent either on the amount of moisture or on certain conditions of freezing, and on the differences in the mineralogical composition.

This method makes it possible to observe alterations in individual inclusions (bubbles of moisture, *etc.*); the old method did not permit of this, thus by using the X-ray method a number of theoretical assumptions can be verified, concerning the causes responsible for the migration (theory of Teber, Fedossov, *etc.*).

The first experiments of the present authors permit to assert that in definite grounds in which the migration proceeds after the type of clay, under certain conditions (we can speak of the conditions of our experiment only) the water drops are not embryos from which ice intercalations are formed.

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Received
30. V. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*ECOLOGY*

# EGG LAYING IN *DROSOPHILA MELANOGASTER* AS INFLUENCED BY SUGAR CONTENT IN THE FOOD

By R. A. MASING

*(Communicated by L. A. Orbeli, Member of the Academy, 24.V.1944)*

In spite of the wealth of work devoted to the study of the insect instinct, the variability and still more the inheritance of the instinct behaviour are far from being sufficiently studied.

Many insects, inclusive of *Drosophila melanogaster* flies, lay their eggs chiefly on the substances that subsequently serve as nutrient medium for the flies themselves and for their offspring as well. Loeb considers this behaviour as being a typical instinct. Our observations on females of certain *Drosophila melanogaster* stocks showed them to choose the medium for egg laying provided there is a slightest possibility of choice. The choice of a medium for oviposition may be considered as an instinct behaviour. A detailed examination of the latter is of a great theoretical interest, the more so as it can easily be estimated in simple quantitative terms.

The present communication contains data on the influence of sugar content in nutrient medium upon oviposition in *Drosophila melanogaster* females*.

The usual yeast nutrient medium with or without sugar was used for the experiments **. The medium to be studied was spread over cover glasses. The latter were placed in Petri dishes. Each Petri dish was made to contain eight cover glasses, four of which were covered with nutrient medium, and four without sugar. The first group of glasses was denoted by +, and the second by −. 15—20 females were put into each Petri dish and kept there during 4—7 hours. The females were then removed and the eggs counted.

For the experiments were used the wild-type flies of the so-called «Florida» strain inbred during 23 generations, and mutants: white eye colour and sepia eye colour strains, and dumpy wing strain of *Drosophila melanogaster*. Preliminary experiments showed that the wild type females as well as the white and sepia females laid their eggs chiefly on medium containing no sugar, while the dumpy females laid the eggs on the sugary food.

To answer the question, what is the most favourable medium for the development of flies, cover glasses containing the eggs laid were transferred into flasks containing food with a high concentration of sugar (25 per cent, against 5 per cent in the standard nutrient medium), and medium containing no sugar. The

* In his study on the influence of different chemical substances upon the laying of the eggs in *Drosophila* Edward Adolph [1] used water solutions of substances without preparing solid substratum. This was probably the cause of a very low number of the eggs laid in his experiments, and this makes the whole results rather questionable.

** Water, 250 g; wheat meal, 15 g; raisins, 12.5 g; sugar, 15 g; yeast, 4 g; agar-agar, 2 g.

latter was smeared with usual water yeast suspension. The non-sugary food is soon covered with mould, and the emerging flies perish soon. Accordingly, the numbers of pupae developed rather than the number of the flies emerged were taken to express the number of the flies developed. The results of the experiments with the wild type flies from the «Kuibyshev» population and those with the dumpy flies are given in the table. They showed that the development of flies proceeds at the same rate in both media under consideration.

Development of Flies in Sugary and Non-sugary Food

| Strains | Non-sugary food | | | Sugary food (25% of sugar) | | |
|---|---|---|---|---|---|---|
| | Number of eggs | Number of pupae | Per cent of pupae developed | Number of eggs | Number of pupae | Per cent of pupae developed |
| «Kuibyshev» .. | 926 | 858 | 93 | 869 | 806 | 93 |
| Dumpy . . . . | 698 | 380 | 54 | 612 | 329 | 54 |

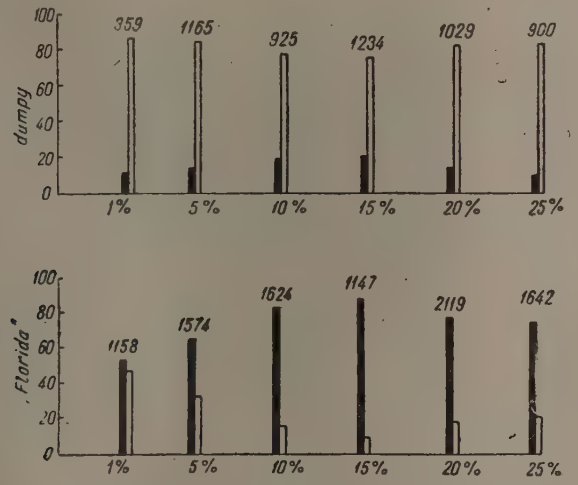


Fig. 1. Ordinate, per cent of eggs laid. Stippled columns, sugary food; white columns, non-sugary food. Abscissa, sugar content in the food. Number over the columns, numbers of the eggs laid

The next step was to determine the response of the flies to the different concentrations of sugar in the media. The experiments were carried out with the wild type flies from «Florida» inbred stock and dumpy flies. To estimate the influence of separate components, raisins and yeast were removed from the medium*. The results of the experiments are given in Fig. 1. There is a strong difference between the stocks compared as to their response to the sugar content in the medium. It should also be pointed out that the flies' behaviour and especially that of the dumpy flies was but little affected by the difference in the sugar content. The pecent of eggs deposited on the sugary food was approximately the same, while the concentration of the sugar was 1 or 25 per cent. It was rather difficult to estimate

* Accordingly, the ingredients taken and their ratio were the following: (1) water, 250 g; agar-agar, 2 g; meal, 15 g; (2) water, 250 g; agar-agar, 2 g; meal, 15 g; sugar.

the lower limit of susceptibility to sugar in the dumpy females, since 85 per cent of the eggs were laid on a medium containing sugar, the concentration of which was equal to 0.2 per cent. «Florida» females were found to be less susceptible to the different sugar content and to fail to discriminate between sugary and non-sugary food when the concentration of it was equal to 1 per cent. Slight variation in the per cent of eggs laid by the females of either line (for example, a well pronounced preference of oviposition in the «Florida» females to the non-sugary food when the sugar medium contained 15 per cent of it) is probably of accidental nature.

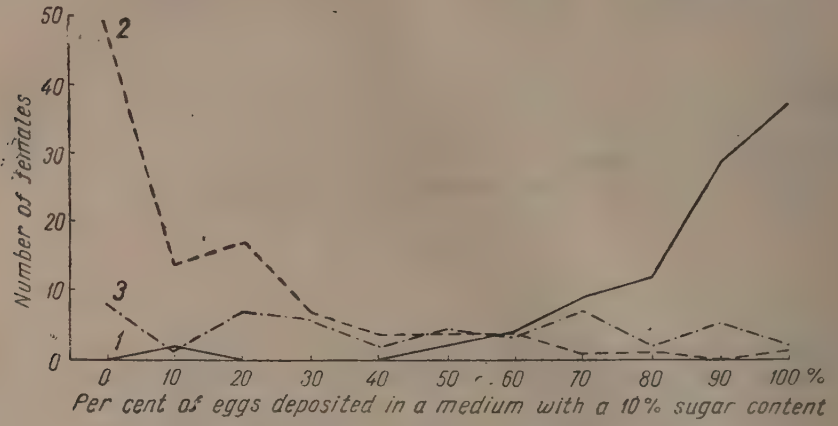


Fig. 2. *1*, dumpy; *2*, «Kuibyshev»; 3, «Florida», inbred stock

In order to make the experimental conditions still simpler, it was desirable to eliminate from the food the wheat meal which is a compound component and gives some dextrin upon boiling. Consequently, the concentration of the components used were as follows: (1) water, 200 g; agar-agar, 2 g; (2) water, 200 g; agar-agar, 2 g and sugar.

As females after being placed into an empty Petri dish failed to lay eggs at all, it might be supposed that such a medium would inhibit oviposition. The experiments showed, however, that the medium under consideration did not affect the intensity of oviposition. The dumpy females were found to respond in the same way as in the preceding experiment. The per cent of the eggs deposited on sugary food was 90, in spite of the fact that the media under comparison contained 1 and 35 per cent of sugar, respectively. The response of «Florida» females to the presence of sugar was not so clearly pronounced as in the preceding experiment. The number of the eggs laid on sugary and non-sugary media was even somewhat higher when the sugar content equaled 35 per cent. It may be supposed that the change in the females behaviour was due to a change in the food composition. The food already tested (with meal, yeast and raisins, respectively) was repeatedly used in a new series of experiments. The sugar content used (10 or 15 per cent) was the one that gave the most distinct results. Nevertheless, the results observed in all the subsequent experiments varied greatly.

As it was already mentioned, each Petri dish was made to contain 15—20 females. As in every experiment a certain part of the eggs only was laid on the foods compared, the results observed for each Petri dish are only some averaged value, characterizing the whole group of the females in general. Therefore, the question arises as to whether the data obtained are the resultant of a more or less similar behaviour of all the females of a given group, or whether they are due to a more considerable deviation with respect to the peculiarity considered.

The «Florida» strain used was highly inbred (33 generations at the beginning of these experiments). It seemed scarcely probable that the variation observed was due to heterozygosity of population used. Nevertheless, 51 females from the inbred «Florida» strain, 29 females from non-inbred «Florida» strain, 101 females from the

wild type «Kuibyshev» strain, 50 females from wild-type «Simferopol» strain, and 93 dumpy females were subjected to individual analysis*. The only females that laid no less than 15 eggs (during 14—16 hours) were taken for the further analysis. The sugary food contained 15 per cent of sugar. The results obtained are given in Fig. 2.

Individual analysis of the character in question showed a high variability of this character in the females of the «Florida» strain. Some of them laid their eggs on the sugary food only, while others prefered the non-sugary food. All the intermediate forms were also present, their frequency being more or less uniform throughout the curve.

The behaviour of females from the «Kuibyshev» and «Simferopol» strains was found to be similar to that of the wild-type inbred «Florida» females, observed in the preceding experiments. The majority of the females laid eggs on the non-sugary food, and only individual flies were found to prefer the sugar containing food. For example, one female from the «Kuibyshev» strain laid eggs only on sugary food. The flies hatched from the eggs were individually tested and found to deposit their eggs preferably on non-sugary food, as it is characteristic of the females from the «Kuibyshev» strain.

Similar individual analysis showed that among the dumpy females there were found flies whose behaviour was unusual for this strain. For example, 2 females laid all their eggs on non-sugary food, while the other two females laid an equal number of eggs, both on sugary and non-sugary food. However, the majority of the females deposited 90—100 per cent of their eggs on sugary medium.

As the variation in the behaviour of «Florida» females was difficult to interpret, the experiments were continued, and individual analyses of the females from «Florida», «Kuibyshev», and dumpy strains were carried out from time to time.

The dumpy females were found to show the highest stability in their preference of sugary food. In the «Florida» strain, in contrast to the first series of experiments, the number of the females that laid their eggs on the sugary medium showed a steady increase. The wild-type «Kuibyshev» females showed a great stability in laying their eggs on non-sugary medium since December, 1940 (when the experiments were started) till April, 1941 (*i. e.* during 7—8 generations approximately), when the number of females showing different types of behaviour gradually increased. Unfortunately, however, all these stocks died. The investigation of other strains, the results of which will be described in another paper, gives some reasons to believe that the change in the behaviour of the wild-type females are due to heterozygosity of the stock.

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and Pathology of Higher Nervous Activity.

Received
24. V. 1944.

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* The data for the females from non-inbred «Florida» strain are not reported in this paper, since the behaviour of these females was very similar to that of the females from inbred «Florida» strain. The same holds true for the females from «Simferopol» and «Kuibyshev» strains, the curves for which are found to coincide with each other.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*GENETICS*

# AN INTERGENERIC HYBRID BETWEEN BARLEY AND *ELYMUS*

By F. Kh. BAKHTEEV and E. M. DAREVSKAYA

*(Communicated by N. V. Zyzin, Member of the Academy, 3.VIII.1944)*

In connexion with the works by N. V. Zyzin (1) remote crosses between cultivated and wild plants were practised on a rather large scale both in this country and abroad. At the Institute of Grain Husbandry, headed by N. V. Zyzin, Member of the Academy, aside from wheat—couch-grass, there are also grown hybrids between rye and couch-grass, between wheat and *Elymus*, between barley and critesion, *etc.* Finally, at the same Institute we have produced for the first time in the history of cereal breeding, the hybrid between cultivated barley and *Elymus*.

It is well-known that all efforts to cross cultivated barley with *Elymus*, either in this country or abroad, remained unsuccessful as yet (3).

The production of hybrids between barley and *Elymus* is, however, of extreme importance for plant breeding, both practically and theoretically.

In the summer of 1943, at the suggestion of N. V. Zyzin we have made crosses between cultivated barley and two *Elymus* species, *E. arenarius* L. and *E. giganteus* Vahl. The genera *Hordeum* and *Elymus* belong to the tribe *Frumentaceae* Andul. of the family *Gramineae* (2).

In the crossings of two-rowed barley, varieties *nutans* 8925 from Portugal and bare-grained «Kolkhoz-7» (v. *nudum* L.), with *Elymus arenarius* no grains were obtained; only from pollination of 44 glowers of two-rowed barley variety «Wiener» (v. *nutans* Schübl.) with the pollen of *E. giganteus* there was obtained a single grain. In the beginning of February 1944 this grain produced a plant which differed from the maternal form of barley. It possessed narrower and coarser leaf blades of a deeper green and was distinguished by the presence of anthocyanine at the base of the young plant.

During the first five weeks, *i. e.* until the beginning of tillering, the development of the plant proceeded normally although it was somewhat delayed. Beginning with this moment, vigorous growth and profuse tillering set in. It proved therefore possible to separate from the plant several rooted stems and plant them into separate pots.

At this stage of development our hybrid displayed the dominance of the general habitus of the paternal plant, *i. e.* of *Elymus*: coarse long leaves, coarse, rather thick and strong stems, a strong development of foliage and a developmental rhythm which was clearly of the *Elymus* type.

Cytological studies made by V. A. Arnoldi, member of the staff of this Institute, have shown that the hybrid possessed 21 somatic chromosomes. This undoubtedly proves that the plant is a real hybrid, the haploid numbers of barley and *E. giganteus* being 7 and 14, respectively.

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Received
3. VIII. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*MICROBIOLOGY*

# ON THE SALT RESISTANCE OF *AZOTOBACTER*

By A. R. WERNER

*(Communicated by A. A. Richter, Member of the Academy, 3.VIII.1944)*

The activity of *Azotobacter* is influenced greatly by the conditions prevailing in the soil. Various odd data ([3–5]) on *Azotobacter* records in different soils show that, unlike higher plants, soils high in salts may offer a favourable medium for these organisms; quite often solonchaks (salines) and solonetz (alkali soils), but poorly suitable for cultivated plants because of their high salt content, show an intense activity of *Azotobacter*. This induced us undertake a comparative study of *Azotobacter* activity in soils varying as to their salt content, reported in the present note.

We studied the soil varieties from the experiment plots of the Ubinsk Experiment Meliorative Station, representing sharply pronounced complexes characteristic of the soils of Southern and Middle Baraba (Western Siberia), which have not been investigated earlier for the presence of *Azotobacter*. The characteristics of the salt regime of the investigated soils are summed up in Table 1. This table is based on current analyses of soil water extracts made at the laboratory of the Station under the guidance of N. V. Orlovsky for several years. The table shows the maximum concentrations of salts recorded within the last 2—3 years.

The activity of *Azotobacter* was evaluated in soil specimens obtained during summer. The method used was by laying out small soil clots on agar plates with Ashby's medium. The use of agar instead of silica gel warranted adequate selectivity of the medium, and 4—5 days after the seeding quite typical colonies of *Azotobacter* would already grow around the clots of those specimens which showed the presence of this organism. Simultaneously with this soil-clot test which served for quantitative counts, individual soils were tested also by making spontaneous soil cultures (according to the method of Winogradsky) and by seeding the soil into a liquid medium (according to the method of Bejerink). Only in the case of a negative result of all the three tests the soils were considered as not containing *Azotobacter*. The results of these tests are summed in Table 2.

The data of Table 2 demonstrate great differences between individual soil varieties: some of them were found to lack *Azotobacter*, while others showed a high activity of this microorganism. Comparing the data concerning *Azotobacter* with those on the salt regime of the soils, one can see that the unsalinized chernozem-meadow soil and the slightly salinized steppifying medium-columnar solonetz (alkali soil) do not contain or are very poor in *Azotobacter*, while all the other soil varieties of both the alkali and the saline series are rich in it.

It is characteristic that the activity of *Azotobacter* varies with salinization, its maximum coinciding with its relatively high degree, after which this activity drops abruptly with the further increase of salt concentration. This is most conspicuous in the soils of the saline series of which the meadow, highly solonchak variety (section No. 15) shows the maximum *Azotobacter* count; while the greatly resem-

Table 1

## Salt Regime of the Soils of Ubinsk Experiment Meliorative Station
(in per cent and m.-eq. per 100 g of absolutely dry soil)

| No. of section / Depth (cm) | Dry residue | | Ignited residue | | $HCO_3$ | | | | $CO_3$ | | | | $SO_4$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1—15 | 15—30 | 1—15 | 15—30 | 1—15 | | 15—30 | | 1—15 | | 15—30 | | 1—15 | | 15—30 | |
| | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. |
| 11 | 0.123 | 0.140 | 0.080 | 0.057 | 0.064 | 1.05 | 0.063 | 1.03 | none | none | none | none | 0.046 | 0.96 | 0.090 | 1.87 |
| 1 | 0.246 | 0.882 | 0.156 | 0.506 | 0.091 | 1.49 | 0.198 | 3.25 | » | » | 0.035 | 1.15 | 0.054 | 1.12 | 0.075 | 7.56 |
| 3 | 0.334 | 0.506 | 0.229 | 0.312 | 0.103 | 1.69 | 0.294 | 2.82 | » | » | 0.070 | 2.30 | 0.054 | 1.12 | 0.057 | 1.40 |
| 13 | 0.404 | 0.672 | 0.217 | 0.448 | 0.234 | 3.84 | 0.200 | 3.28 | traces | — | 0.008 | 0.26 | 0.067 | 1.40 | 0.107 | 2.20 |
| 8 | 0.554 | 0.674 | 0.370 | 0.438 | 0.327 | 5.36 | 0.325 | 5.33 | » | — | 0.039 | 1.28 | 0.164 | 3.41 | 0.186 | 3.87 |
| 15 | 1.244 | 1.383 | 0.822 | 0.747 | 0.079 | 1.30 | 0.073 | 1.20 | none | none | none | none | 0.271 | 5,64 | 0.277 | 5.77 |
| 16 | 1.814 | 4.018 | 3.143 | 3.007 | 0.062 | 1.02 | 0.060 | 0.99 | » | » | » | » | 1.112 | 23.15 | 0.973 | 20.20 |

| No. of section / Depth (cm) | Cl | | | | Ca | | | | Mg | | | | $Na$ by difference | | Sum total of anions / Sum total of cations | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1—15 | | 15—30 | | 1—15 | | 15—30 | | 1—15 | | 15—30 | | 1—15 | 15—30 | 1—15 | 15—30 |
| | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. | % | m.-eq. | m.-eq. | m.-eq. | m.-eq. | m.-eq. |
| 11 | 0.021 | 0.59 | 0.004 | 0.11 | 0.021 | 1.05 | 0.020 | 1.00 | 0.022 | 1.81 | 0.013 | 1.07 | none | 0.94 | 2.60 / 2.86 | 3.06 / 3.01 |
| 1 | 0.010 | 0.28 | 0.009 | 0.25 | 0.022 | 1.10 | 0.023 | 1.15 | 0.011 | 0.92 | 0.013 | 1.07 | 0.87 | 2.84 | 2.89 / 2.89 | 5.06 / 5.06 |
| 3 | 0.017 | 0.48 | 0.010 | 0.28 | 0.028 | 1.40 | 0.039 | 1.95 | 0.014 | 1.15 | 0.011 | 0.92 | 0.74 | 3.68 | 3.29 / 3.29 | 4.50 / 6.55 |
| 13 | 0.069 | 1.95 | 0.023 | 0.65 | 0.019 | 0.95 | 0.015 | 0.75 | 0.012 | 0.99 | 0.016 | 1.32 | 5.25 | 4.06 | 7.19 / 7.19 | 6.13 / 6.13 |
| 8 | 0.015 | 0.42 | 0.016 | 0.45 | 0.030 | 1.50 | 0.019 | 0.95 | 0.011 | 0,92 | 0.011 | 0.92 | 7.23 | 7.78 | 9.19 / 9.65 | 9.65 / 9.65 |
| 15 | 0.384 | 10.94 | 0.400 | 11.30 | 0.053 | 3.65 | 0.046 | 2.30 | 0.085 | 6,90 | 0.102 | 8.39 | 8.14 | 7.58 | 17.88 / 18.69 | 18.27 / 18.27 |
| 16 | 1.285 | 36.26 | 5.979 | 27.63 | 0.289 | 14.43 | 0.212 | 10.59 | 0.212 | 17.44 | 0.151 | 12.42 | 28.56 | 25.81 | 60.43 / 60.43 | 48.82 / 48.82 |

Table 2

Activity of *Azotobacter* in Soils of the Ubinsk Experiment Meliorative Station

| No. of section | Soil | *Azotobacter* counts at different depths in % of soil clots giving rise to growth | | |
|---|---|---|---|---|
| | | 1—5 | 10—15 | 20—30 |
| 11 | Meadow chernozem | 0 | 0 | 0 |
| 1 | Steppifying medium-columnar solonetz (alkali soil) | 3 | 0 | 0 |
| 3 | Meadow strongly alkaline, strongly saline soil | 72 | 81 | 79 |
| 13 | Meadow strongly solonchakous (saline) solonetz (alkali soil), approaching soil No. 3 | 69 | 73 | 57 |
| 8 | Meadow high-columnar solonetz-solonchak (alkali-saline) | 82 | 14 | 3 |
| 7 | Meadow-swamp solonchakous (saline) soil | 41 | 68 | 35 |
| 15 | Meadow strongly solonchakous (saline) soil | 54 | 71 | 19 |
| 16 | Meadow solonchak (saline) | 12 | 2 | 0 |

bling, but more salinized meadow solonchak (section No. 16) is definitely poor in *Azotobacter*; in the latter soil we have evidently to deal with the limit salinity for *Azotobacter*. In the investigated soil varieties of the solonetz (alkali) series this maximum does not become apparent, and it is to be assumed that among the alkali soils of the investigated range there are none of such salinity as would put a limit to the activity of *Azotobacter*.

Now, if we are to compare the activity of *Azotobacter* with the qualitative salt composition of the salinized soils, it is impossible to establish what is the influence of a given type of salinization: sodium-sulphate salinized soils of the alkali series (sections 3, 13 and 8) and sulphate-chloride salines (sections 7, 15 and 16) are equally rich in *Azotobacter*, in spite of their different chemical and physical properties. This speaks against the assumptions of Germanov ([2]), in whose opinion the high activity of *Azotobacter* in salines and alkali soils and its activity in chernozem and solonized soils should be correlated with the humate part of their replaceable complex. Should the activity of *Azotobacter* in salinized soils depend on energetic substance, as it is assumed by Germanov, there would exist a definite difference between saline and alkali soils. Yet, as this is easily gathered from our data, the investigated solonchaks and solonetz soils are equally rich in *Azotobacter*, and only the concentration of salts forms the decisive factor for this activity.

Considering the degree of salinity necessary for *Azotobacter*, which reaches concentrations as high as 0.5—1.0 per cent, one should class it among halophyte organisms, and regard the action of salts on it as that of stimulants. This stimulating effect of salts we are inclined to correlate with transformations of hormone-like substances within the cells and with the need of *Azotobacter* in growth substances of the «bios» type ([1]).

Ubinsk Experiment
Meliorative Station.

Received
3. VIII. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*PLANT PHYSIOLOGY*

# RAISING THE RESISTANCE OF GRASSES TO RUST (*PUCCINIA*)

By K. SUKHORUKOV and O. SMIRNOVA

*(Communicated by A. A. Richter, Member of the Academy, 3.VII.1944)*

The resistance of plants to rust is noticeably influenced by the external agents and conditions of mineral nutrition. However, the physiological foundations of resistance remain obscure as yet. Sukhorukov and Ovcharov (5) studied the chemical peculiarities in wheats differing in susceptibility to rust (*Puccinia triticina*).

They have found a greater ammonium content in the leaves of plants possessing greater resistance. In plants infected by rust Richter and Grechushnikov (4), using a biological method, have demonstrated the presence of toxic substances which are products of the fungus life-activity. The composition of the excreta of the rust proved to be rather simple and common to many fungal species: they consist of ammonia, urea and thiourea (3). Comparing the results of the studies, the regularities of development and of geographic distribution (1), one is led to a conclusion that the possibility of the development of the fungus within a plant body depends on the plant's nitrogen exchange, on the rate of assimilation of the products excreted by the fungus, and the quantity of ammonia present in the cells.

In order to check up the above conclusion and the results obtained by Sukhorukov and Ovcharov (5) experiments were arranged on introducing ammonia into the plant not *via* roots followed by inoculation with rust. The experiments were carried out under greenhouse conditions at the Tomsk station for plant protection. As a material served winter-wheat (*Triticum vulgare* v. *lutescens* 0329) and winter-rye (*Secale cereale*, variety Vyatka). The two plants in all the experiments were tested at the tillering stage; they were grown in pots filled with soil and kept under optimal moisture conditions. Inoculation was done with uredospores. Wheat was inoculated with *Puccinia triticina* Eriks. collected on winter-wheat and with *P. persistens* Plow from couch-grass (*Agropyrum repens*) in the vicinity of Tomsk. *P. persistens* readily migrates from the couch to wheat. Rye was inoculated with *P. graminis* Pers. from couch-grass.

The experiments were arranged in three replicates with 5 plants in each. The following chemically pure nitrogen compounds were used: monometallic phosphoric ammonia and urea. The experimental plants were sprinkled by means of a pulveriser with these substances in the form of 0.5, 1.0 and 3.0 per cent solutions. Pulverisation was always done in the evening and repeated 10—12 hours later. In the morning inoculation was practised also by means of a pulveriser with a suspension of spores. The period of incubation for the rusts in question (*Puccinia persistens*, *P. triticina*, *P. graminis*) ranges from 8 to 10 days. The intensity of infection was estimated on the 15th day by counting the pustulae on all the leaves; for each kind of treatment 15 plants (3 replicates, 5 plants in each) were recorded.

In Table 1 are summarized the average results for an individual plant.

It may be seen from the table that the introduction of ammonia and urea has in all cases raised the resistance of wheat to *P. persistens*. *P. persistens*, judging from the number of pustulae on the inoculated plants, has shown lesser aggressiveness than *P. triticina*.

Table 1

| Substances applied | Winter-wheat + *P. persistens* (from couch-grass), inoculated 19.VII, observed 4.VIII | | | Winter-wheat + *P. persistens* (from couch-grass), inoculated 12.IX, observed 27. IX | | | Winter-wheat + *P. triticina* (from wheat), inoculated 2. VIII, observed 17. VIII | | |
|---|---|---|---|---|---|---|---|---|---|
| | Concentration, in % | Number of pustulae per plant | Intensity of infection (in % to control) | Concentration, in % | Number of pustulae per plant | Intensity of infection (in % to control) | Concentration, in % | Number of pustulae per plant | Intensity of infection (in % to control) |
| Control | 0 (water) | 68 | 100.0 | 0 (water) | 32 | 100.0 | 0 (water) | 142 | 100.0 |
| $NH_4H_2PO_4$ | 0.5 | 45 | 66.2 | 0.5 | 14 | 43.8 | 0.5 | 196 | 138.0 |
| » | 1 | 30 | 44.1 | 1 | 14 | 43.8 | 1 | 127 | 90.0 |
| » | 3 | 16 | 23.5 | 3 | 25 | 78.1 | 3 | 105 | 73.2 |
| $CO(NH_2)_2$ (urea) | 0.5 | 14 | 20.6 | 0.5 | 12 | 37.5 | 1 | 236 | 166.2 |
| Do | 1 | 42 | 61.8 | 1 | 17 | 53.1 | 3 | 195 | 137.3 |
| » | 3 | 10 | 14.7 | 3 | 14 | 43.8 | — | — | — |

In the case of inoculation with *P. triticina* ammonium also yielded a positive result, except for the 0.5 per cent concentration. With urea, on the contrary, the resistance was even clearly reduced. The amount of substance applied was evidently insufficient to arrest the development of this rust. It seems not improbable that there exists a connexion between the different degree of aggressiveness of the rusts and the different resistance to the ultimate product of their nitrogen exchange and to the presence of ammonia in the inoculated plant. The results of the experiments with inoculation of winter-rye are summarized in Table 2.

Table 2

| Substances applied | Winter-rye + *P. graminis* (from couch-grass), inoculated 19. VII, observed 4.VIII | | | Winter-rye + *P. graminis* (from couch-grass), inoculated 1.X, observed 15.X | | |
|---|---|---|---|---|---|---|
| | Concentration, in % | Number of pustulae per plant | Intensity of infection (in % to control) | Concentration, in % | Number of pustulae per plant | Intensity of infection (in % to control) |
| Control | 0 (water) | 95 | 100.0 | 0 (water) | 7 | 100.0 |
| $NH_4H_2PO_4$ | 0.5 | 51 | 43.2 | 1 | 4 | 57.1 |
| » | 1 | 33 | 34.7 | 3 | 3 | 42.9 |
| » | 3 | 49 | 51.6 | 0.5 | 4 | 57.1 |
| $CO(NH_2)_2$ (urea) | 0.5 | 72 | 75.9 | — | — | — |
| Do | 1 | 54 | 56.8 | 1 | 10 | 142.9 |
| » | 3 | 31 | 32.6 | 3 | 4 | 57.1 |

Under the application of ammonia and urea winter-rye has noticeably raised its resistance to *Puccinia graminis*. An exception to the general rule present the experiments with 1 per cent urea, inoculation done on 1. X. A sharp drop in the aggressiveness of the rust was observed in this series, which should evidently be connected with the changes in the light and temperature conditions in the greenhouse.

The results of our experiments justify the following conclusion. The resistance of grasses to rust can be increased by introducing ammonia and urea otherwise than *cia* roots. A sharp increase in resistance is observed in the case of the forms of rust showing a low aggressiveness; the increase is less conspicuous in the case of aggressive forms. Immunity and resistance to rust are evidently based upon some physiological peculiarities of the nitrogen exchange of the infected plants.

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Received
3. VIII. 1944.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*EXPERIMENTAL MORPHOLOGY*

# INDUCTIVE INFLUENCE OF FORE-BRAIN UPON BODY EPITHELIUM IN *AMPHIBIA*

By F. N. KUCHEROVA

*(Communicated by I. I. Schmalhausen, Member of the Academy, 1.XI.1944)*

Repeated attempts made by several authors to remove the olfactory placode in *Anura* embryos failed to give desirable results. In all these experiments the development of the placode was obtained in embryos in which it had previously been removed. Such a behaviour of the olfactory placode seemed to reduce the value of experimentation, since a number of processes connected with the morphogenetic influence of the olfactory placode could not be studied under experimental conditions.

In their experiments on the removal of the olfactory placode in *Anura* embryos Bell ([2]), Burre ([1]) and Kucherova ([5]) have observed the development of the placode after its being removed in an embryo. The conclusion arrived at by Bell and Burre was that the olfactory placode in *Anura* is endowed with a great regenerative power which secures the regeneration of the placode after its removal. Zwilling ([3]) observed the development of the olfactory placode on the side where the original placode had previously been removed. Zwilling's explanation of this experiment was that the fore-brain had induced a new material that replaced the olfactory placode removed. Lopashov ([4]) observed that the removal of the ectoderm in amphibian embryos at the late gastrula stage resulted in a lack of development of the nervous system and in the absence of the buds like those for the otic vesicle and olfactory placode. Similar results have been obtained by Krivonossov in his experiments on the removal of the neural fold in *Rana esculenta* embryos at the neurula stage. The conclusion arrived at by these authors was that both these organs were induced by the corresponding regions of the neural tube. Thus, the question as to the development of the olfactory placode after its being removed in *Anura* embryos is subject to some controversy. Kogan ([6]) studied the possibility of inductive influence of the medulla oblongata on the body epithelium in the embryos at late gastrula and neurula stages. It was found that the medulla oblongata being transplanted beneath the abdomen epithelium causes a development of the otic vesicle identical to the normal one.

The purpose of the present paper was to study the ability of the fore-brain to induce epithelium. With this purpose in mind two series of experiments were performed on *Anura* embryos, in which the transplantation of the fore-brain beneath the side epithelium of an embryo and the transplantation of the abdomen epithelium in the place of the olfactory placode removed, were carried out. In the latter case for the determination of the exact place of induction the epithelium transplanted to the place of the placode removed had previously been marked by the carmine grains. The theoretical grounds taken as a basis for the experiments were as fol-

lows. If the fore-brain produces an inductive influence on the epithelium in the course of the normal development of the olfactory placode in *Anura* embryos, the piece of the fore-brain being transplanted beneath the abdomen or side epithelium at the early neurula stage must induce in the latter the development of an olfactory placode, just like the development of the otic vesicle in the abdomen epithelium beneath which the piece of the medulla oblongata had been transplanted. Besides, the inductive power of the fore-brain can be tested by means of transplantation of the abdomen epithelium in place of the olfactory placode removed. The experiments were carried out on *Rana esculenta* and *Bufo viridis*.

Two series of experiments were performed: (1) transplantation of a piece of the fore-brain from an embryo at the early caudal bud stage; 2) transplantation of the abdomen epithelium previously marked in place of the olfactory placode removed. 81 operations were done in the first series of experiments. The embryos were fixed with $HgCl_2$ and acetic acid 4, 6 and 9 days after operation, and stained with boric carmine. The sections done were 10 μ thick. The second series of the experiments comprised 47 embryos operated at the stage of the early caudal bud. The fixed material was stained with picric acid to prevent the masking of the transplant that had been marked with carmine.

The results of the first series of the experiments showed that all the embryos contained their transplants as a piece of the neural tissue having an inner cavity resembling that of the neural tube. In the epithelium of the embryo skin opposite the transplant the olfactory placode was developed, the form and histological differentiation of which looked like those of a normal placode. It consisted in the invaginating skin epithelium with several layers of cells. Microscopic examination of the material showed that the inductive power of the fore-brain in the embryos of the first term of fixation was not yet definitely manifest, and its influence on the body epithelium was insufficient, while six days after operation it was completely realized and caused a development of the olfactory placode in the epithelium of the embryo skin. The results of the second series of the experiments comprising transplantations of the abdomen epithelium to the olfactory placode removed showed that the olfactory placode induced was always developed from the epithelium transplanted. This was shown by the presence of the carmine grains in the epithelial cells of the transplant. The placode induced was identical with the normal one.

Thus, the results of the experiments described showed that the fore-brain is able to induce the olfactory placode in the abdomen epithelium transplanted in place of the olfactory placode previously removed. The investigation of the embryos at later stages of development showed that the subsequent development of the placode is brought about in a normal way. However, it must be pointed out that the subsequent development of the placodes induced in the embryos of the first experimental series comprising the transplantation of a piece of fore-brain beneath the abdomen epithelium was considerably retarded.

In the literature devoted to the induction of the olfactory placode there are known a few number of investigations whose authors tried to make clear the relations between the fore-brain and the olfactory placode in the development of the olfactory organ. The conclusions arrived at by the authors are rather conflicting. Bell asserts that the olfactory placode is being developed independently from the brain. Transplanting a part of the brain corresponding to the olfactory lobe into another part of the body of *Rana esculenta* embryo (over the optic vesicle) Bell (2) failed to observe the induction of the olfactory placode. He concluded therefore that the placode is developed independently from the brain. On the other hand, this author observed the development of the placode in *Rana esculenta* embryo on the side where the original placode had previously been removed. The conclusion made by him from this experiment was that the placode might regenerate after its removal. The same conclusion has been arrived at by Burre (1) who observed the development of the placode in frog's embryos after its removal. The opposite point of view was expressed by Zwilling who asserted that the olfactory placode is developed in the close dependence upon the fore-brain. He was accidentally

successful in obtaining the induction of the olfactory placode in his experiments on transplantation of the optic vesicle (that evidently included the material of the fore-brain) to embryos of *Rana pipiens*. To re-examine this question Zwilling carried out the experiments on *Amblystoma punctatum* and observed the secondary induction of the olfactory placode at the place where this rudiment had previously been removed. He came to the conclusion that the development of the secondary placode was due to the fore-brain induction.

In my experiments on transplantation of epithelium in *Rana esculenta* and *Bufo viridis* embryos I was successful in determining with the highest precision the place of induction, for the olfactory placode developed from the very epithelium transplanted to the place of the placode removed. This could be seen from the presence of carmine grains in the transplant epithelium. Thus, the data obtained in my experiments went to confirm the inductive influence of the fore-brain upon the body epithelium in *Anura* embryos. In addition, in the experiments on transplantation of the fore-brain piece beneath the abdomen epithelium I have observed the development of the olfactory placode from the epithelium beneath which the fore-brain piece had been implanted. In this place there was observed in the epithelium typical invagination characteristic of the development of the olfactory placode. The results of my experiments do not confirm the conclusions of Bell who, implanting the fore-brain piece beneath the epithelium in *Rana esculenta* embryos, did not observe the induction of the olfactory placode. My experiments showed the development of the rudiment in question in all the cases, its historical differentiation being typical and often normal in shape. It should be pointed out, however, that the further development of the placode induced in the abdomen epithelium under the influence of the piece of the fore-brain implanted was retarded.

Conclusions. 1. The fore-brain fragment corresponding to the olfactory lobe of the *Rana esculenta* embryo at a stage of early caudal bud being transplanted at the early neurula stage beneath the abdomen epithelium induced in the latter the development of the olfactory placode.

2. In the abdomen epithelium taken from *Rana esculenta* and *Bufo viridis* embryos at the early neurula stage marked by the carmine grains and then transplanted in place of the olfactory placode removed there was observed the development of the placode induced under the influence of the brain. Its further development does not differ from that of a normal placode.

3. The fore-brain of *Rana esculenta* and *Bufo viridis* at the early caudal bud stage possesses the power of inducing the olfactory placode from the abdomen epithelium at the stage of early neurula.

Received
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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1945. Volume XLVII, № 4

*EXPERIMENTAL MORPHOLOGY*

# ON THE COMMON NATURE OF THE EFFECT PRODUCED BY THE EYE IN THE PROCESSES OF DETERMINATION OF LENS OR CORNEA

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*(Communicated by I.I. Schmalhausen, Member of the Academy, 3.X.1944)*

In spite of the fact that the lens and the cornea are the components of the same organ, *viz.* the eye, either of them gave rise to an individual chapter in the science which we call development mechanics. The study of the process of determination of lens and cornea developed according to a common plan, however, which made it possible to compare the separate conclusions inferred from a number of individual experiments. In connexion with the present study, the author was interested in four of them:

I. The development of the cornea in the tadpole does not necessarily imply that the epidermis be acted upon by a whole eye rudiment, but it may also be induced by an eye rudiment deprived of its lens ([1]).

II. The epidermis of a tadpole acted upon by an eye rudiment at different developmental stages transforms into cornea ([2]).

III. The eye rudiment of a tadpole kept at different developmental stages preserves its capacity to induce the transformation of a fragment of epithelium grafted on to its cavity into a lens ([3]).

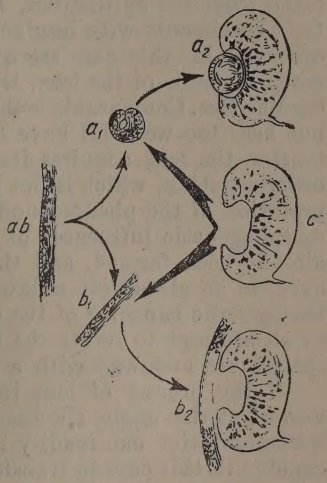
IV. The skin rudiment of a tadpole at different development stages does not lose its lens-forming capacity either ([4]).

All these conclusions, however, have been inferred from experiments using as material various animals at different developmental stages. The methods applied were not the same and, as a matter of course, the results obtained are comparable not without qnalification. Yet a comparison of the phenomena observed in the case of homogeneous material might be interpreted to mean that the eye rudiment of the tadpole at different developmental stages is endowed with a capacity to induce the formation of both the lens and the cornea from a skin rudiment, irrespective of the latter's stage of development. One might thus assume that eye rudiments of the same age are likely to induce from skin rudiments of a similar age either a lens or a cornea. The theoretical import of this assumption made imperative a check up of all the four phenomena referred to above in identical material of similar age, which was performed by the present author. The experiments had to demonstrate that eye rudiments of the same age ($c$) are capable to induce from the very same fragment of skin ($ab$) a lens ($a_2$) when implanted within the eye cavity ($a_1$), or the cornea ($b_2$), in case the fragment is grafted above the eye ($b_1$) (see figure).

Tadpoles of the species *Bufo viridis* were taken as material. According to my personal observations which are in fair agreement with data available in literature ([5]), a feature peculiar to this species is that upon the removal of the lens the eye rudiment proves unable to induce regeneration of a lens from the edge of the

eye. This property is of paramount importance for the experiment which I undertook in order to demonstrate that it was just an eye rudiment deprived of its lens that was capable to induce the formation of the cornea. The operation was carried out at a stage corresponding to that of a tadpole with a scarcely observable limb bud.

The first experiment consisted of two runs. In one run the lens of the eye was removed and in its place was transplanted a fragment of a skin rudiment excised from the head. The very first observations went to show that if transplanted in this way, an eye rudiment protrudes strongly above the level of the epidermis, thus causing a considerable tension of the overlying epidermis. It has been reported that the tension of the epidermis may occasionally be mistaken for the process of cornea induction ([6]). On the other hand, however, data are available according to which the interlayer of connective tissue between the eye and the skin may prevent this skin area from transforming into cornea ([2]), wihch inevitably occurs in the case the eye rudiment is grafted under the abdomen epidermis. It was therefore found advisable to undertake another experiment, consisting of two runs, too. Now both the lens and the cornea were removed in the experimental tadpoles. After 24 hours the epidermis was found to have regenerated, and the defect to be covered with opaque pigmented skin. Part of the tadpoles operated upon were taken for the second run, while the first was made up of such tadpoles as had been grafted with «regenerated» epidermis in place of the lens excised. The identity of age of the experimental material was strictly observed in both runs, the tadpoles being taken from a single hatch. The experimental material was thus absolutely homogeneous in both cases, *viz.* producing either the cornea or the lens when acted upon by eye rudiments identical as to their properties.




Vital observations proved sufficient to show that an eye rudiment deprived of the lens is capable of transforming a given area of the skin rudiment into cornea. The whole process is complete within 8 to 15 days. With special clarity the transformation of the «regenerated» epidermis into cornea was borne out in the 1st run of experiment II: the epidermis which overgrew the defect, intensely pigmented at first, was soon found to transform into transparent cornea, differing in no way from the normal one. The development of the lens could also be observed *in vivo*; the lens induced was either larger or smaller than the control, which seemed to depend on the size of the fragment of epidermis that was implanted. The microscopic picture went to confirm the observational results.

The question presents itself just what is responsible for the fact that the same skin area acted upon by the same eye rudiment gives rise now to a lens and then to cornea? An answer to this question is to be sought in the analysis of the phenomenon of a lens induced from an epithelium fragment grafted on to the cavity of the eye-cup. In a number of studies ([7]) Popov turned to advantage the phenomenon that had been revealed by Manuylova ([3]), *viz.* that the eye rudiment preserves its capacity to induce a lens from an area of a skin rudiment grafted on to the cavity of the cup. The interpretation he gave to the phenomenon discovered was a wrong one: he termed it the «morphogenesis» of the lens, comparing the process observed with the one of regular morphogenesis of the lens formed from the inner layer of the ectoderm under the influence of an eye vesicle. Yet the two processes differ materially. The typical process of lens-formation is naturally subdivided into two stages; the first of these begins with the tension of the lens and lasts until the lens separates from the epithelium; the second consists in the process of differentiation of the lens separated from the epithelium. Unlike this, if a fragment of epithelium

is transplanted into the cavity of the eye rudiment, the first stage of the process is lacking at all, being substituted by the artificial excision of the epithelium fragment; by dint of its inherent properties the latter is within 30 minutes found to be inclosed in a void vesicle both *in vivo* and *in vitro*, which makes it resemble the lens separating by constriction. Thus, the first stage of morphogenesis, which is a rather complicated process, is found to be performed in this case by the experimenter himself. As a matter of fact, in the case of a typical development of the lens, we have to do with histogenesis rather than with morphogenesis; the histogenesis consists in a number of consecutive transformations of the void lens rudiment separated from the epithelium. It is only this part of the process that we observe in the experiments with implanted isolated epithelium areas. Indeed, the phenomena observed in this case are quite incomparable to those accompanying the typical morphogenesis of the lens, which have served as a basis for deducing a number of regularities. Comparable is the second stage of the process of lens development only, but here too we shall have to compare different qualities, for when subjected to tension the lens acquires its specific properties, *viz.* its capacity to further self-differentiation, which is not peculiar to the epithelial vesicle. The fallacious interpretation of the phenomenon discovered gave rise to a search after the «strongest morphogenetic influence» of the eye rudiment, which appears after the lens had already been formed, and this contradict the laws of biology. It gave rise, moreover, to an absolutely unlawful revision of the regularity disclosed in the varying lensogenetic capacity of the epithelium of different amphibian species, as well as to an attempt to refute the fact discovered by me, *viz.* that different amphibian species are endowed with a different inducing power of their eye rudiments ([8]).

If the process of lens induction from an implanted epithelium fragment is viewed at this angle, the ease with which it is induced in the case of intraocular transplantation can readily be accounted for. It is obvious that the epithelium is capable in this case to transform into a lens because it did not lose its capacity to transform into the cornea. And this is also the cause responsible for the development from one and the same epidermis acted upon by the same cup rudiment now of a lens and then of the cornea. It is quite obvious that if a single factor is changed in the otherwise identical conditions of two experiments yielding different results, and this single factor is the position occupied in a given experiment by the skin rudiment exposed to the influence of the eye rudiment, it is this very factor that should be made responsible for the results of the two experiments being at variance. If in the system the skin area is stretched above the eye, it is found to transform into cornea; if it is found to be enclosed in an isolated vesicle, it is transformed into a lens. The conclusion suggests itself that if the same source of morphogenetic influence acting upon the same source of morphogenetic reaction gives rise to the formation of two organs externally differing as to their physical conditions, their shape, the essence of the determining influence operative in the two processes is obviously the same.

The conclusion as to a common nature of the determining influence of the eye rudiment in the processes of induction of both lens and cornea offers a clue to the understanding of more than one phenomenon which still defied interpretation. It may be helpful, moreover, in working out the problem of further investigations of the mechanism and the essence of the determination process.

Received
3.X.1944.

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